New Journal of Physics
The open–access journal for physics

Multipartite entanglement in spin chains

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New Journal of Physics **7** (2005) 229
Received 18 July 2005
Published 4 November 2005
Online at http://www.njp.org/
doi:10.1088/1367-2630/7/1/229

Abstract. We investigate the presence of multipartite entanglement in macroscopic spin chains. We discuss the Heisenberg and the XY model and derive bounds on the internal energy for systems without multipartite entanglement. Based on this we show that in thermal equilibrium the above-mentioned spin systems contain genuine multipartite entanglement, even at finite modest temperatures.

Contents

PII: S1367-2630(05)04003-6

1. Introduction

The study of entanglement in condensed matter systems has become a major line of research in quantum information science. In these studies, different aspects of entanglement have been investigated. Firstly, by investigating the entanglement in the reduced state of two qubits in a spin chain, it has been shown that entanglement theory can help to understand the physical properties of systems undergoing quantum phase transitions [1]. A slightly different approach studies the entanglement between two qubits which can be generated by local operations on the remaining qubits, the so-called localizable entanglement [2]. A third field of study investigates the entanglement between a block and the remaining qubits in the ground state. The entanglement can be quantified by the entropy of the reduced state and then the question arises, whether the entanglement scales like the surface of the block [3]. Furthermore, the insights from entanglement theory made it possible to understand the limits of known simulation techniques (like the density matrix renormalization group) in condensed matter physics as well as the design of new techniques which are superior to the ones known before [4]. Finally, it has been shown that macroscopic properties of solids can be related to entanglement properties of microscopic degrees of freedom, allowing for a detection of entanglement in macroscopic objects by observing macroscopic observables [5, 6][1].

In all of the research lines specified above, entanglement was studied as a *bipartite* phenomenon; the investigation of *multipartite* entanglement in spin chains has only a limited literature [7, 8]. In entanglement theory, however, multipartite entanglement has been intensively studied and it is known that quantum correlations in the multipartite setting have a much richer structure than in the bipartite setting [9, 10]. The main results known are for three or four particles; results for large ensembles of particles are still rare. Nevertheless, studying multipartite entanglement in large systems can already benefit from the results for small systems in two respects. Firstly, on a more local scale, one can pick up sub-ensembles of three or four particles and investigate the multipartite entanglement of the corresponding reduced state. Secondly, on a more global scale, one can ask what types of multipartite entanglement are necessary to form a given state of the total system.

In this paper, we investigate multipartite entanglement in macroscopic spin systems. The tool we use is similar to the one introduced in [5]: we derive bounds for the internal energy $U = \langle H \rangle$ which have to hold for states without multipartite entanglement, violation of these bounds implies the presence of multipartite entanglement in the system. We investigate in detail two one-dimensional spin models: the Heisenberg model and the isotropic XY model. Our results lead to the insight that for these models even at modest temperatures the correlations cannot be explained without assuming the presence of genuine multipartite entanglement. However, we would like to stress that in our approach we do not assume that the spin system is in the ground state or in a thermal state; our theorems can also be applied to states out of equilibrium.

Our paper is organized as follows: In section 2, we introduce the basic definitions of multipartite entanglement we want to apply to spin systems. We first recall the notion n-separability and genuine multipartite entanglement. Then, we also introduce the notion

[1] In these works, the conclusion that the state is entangled relies on the assumption that the model for the physical system is valid. Strictly speaking, one proves the existence of entanglement in a certain model of statistical mechanics.

of k-producibility, which is well suited for the investigation of multipartite entanglement in macroscopic systems. In section 3, we apply these terms to the anti-ferromagnetic Heisenberg chain. We calculate energy bounds below which either reduced states are genuine multipartite entangled or the total state requires multipartite entanglement for its creation. This shows that already at temperatures of $kT \approx J$ the effects of multipartite entanglement cannot be neglected. In section 4, we demonstrate that our proofs also work for other models by calculating similar thresholds for the XY model. Finally, we summarize our results and name some open problems.

2. Notions of multipartite entanglement

Let us start by clarifying the terms we will use to classify multipartite entanglement in spin chains. For a pure state $|\psi\rangle$ of a quantum systems of N parties we may ask for a given $n \leqslant N$, whether it is possible to cluster the parties into n groups, such that $|\psi\rangle$ is a product state with respect to this partition,

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_n\rangle. \tag{1}$$

If this is the case, we call the state $|\psi\rangle$ *n-separable*. A state which is N-separable is a product state with respect to all subsystems, these we call *fully separable*. If a state is not two-seperable (biseperable), we call it *genuine N-partite entangled*. For a mixed state described by a density matrix ϱ these terms can be extended through convex combination. If we can write $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $p_i \geqslant 0$, $\sum_i p_i = 1$ and n-separable $|\psi_i\rangle$ we call ϱ n-separable. Physically, this means that a production of ϱ requires only n-separable pure states and mixing[2].

Besides asking for n-separability, we may also ask questions like 'Do two-party entangled states suffice to create the mixed state ϱ?' This leads to the following definition. We call a state $|\psi\rangle$ *producible by k-party entanglement* (or k-producible, for short) if we can write

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_m\rangle, \tag{2}$$

where the $|\phi_i\rangle$ are states of maximally k parties. So, in this definition $m \geqslant N/k$ has to hold. This definition means that it suffices to generate specific k-party entanglement to arrive at the state $|\psi\rangle$. Conversely, we call a state *containing genuine k-party entanglement* if it is not producible by $(k-1)$-party entanglement. This definition can be extended to mixed states as before via convex combinations. Again, a mixed state which is k-producible requires only the generation of k-party entangled states and mixing for its production. Consequently, a mixed state ϱ contains k-party entanglement, iff the correlations cannot be explained by assuming the presence of $(k-1)$-party entanglement only.

[2] Note that a mixed state $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ can usually be written as a convex combination in many ways. Thus a state which was generated by mixing entangled states might nevertheless be separable, since on may create the same state also by mixing separable states. Note also that for an n-separable state ϱ the states $|\psi_i\rangle$ may be n-separable with respect to different partitions.

Obviously, there are some relations between the notions of k-producibility and n-separability. For instance, the states containing N-party entanglement are just the genuine multipartite entangled states and the one-producible states are the fully separable states. Furthermore, a state of which some reduced state of m parties is genuine m-partite entangled, contains m-partite entanglement, while the converse is in general not true[3].

In the thermodynamic limit $N \gg 1$, however, the notion of n-separability for the total system becomes problematic. This is mainly due to two reasons. Firstly, it does not take into account how many particles are entangled. A biseparable state can be a product state for two partitions of equal size or just one qubit might be separated from a large genuine multipartite entangled state; these cases are not distinguished. Secondly, statements about n-separability require the exact knowledge of N—a precondition which is usually not fulfilled in realistic situations. In stark contrast, the notion of k-producibility is designed to be sensitive to the question 'How many qubits are entangled?' Also, deciding whether a state is k-producible does not require the exact knowledge of N. If one controls only $k \ll N$ subsystems, one may still conclude that the total state contains k-party entanglement, e.g. if the reduced state is genuine k-party entangled.

3. The Heisenberg model

Let us start the discussion of spin models with the anti-ferromagnetic isotropic Heisenberg model. Here, we assume a one-dimensional chain of N spin $1/2$ particles with periodic boundary conditions. The Hamiltonian of this model is given by

$$H_H = J \sum_{i=1}^{N} \sigma_x^{(i)} \sigma_x^{(i+1)} + \sigma_y^{(i)} \sigma_y^{(i+1)} + \sigma_z^{(i)} \sigma_z^{(i+1)}, \tag{3}$$

where $\sigma_x^{(i)}$ denotes the Pauli matrix σ_x, acting on the ith qubit and $J > 0$ the coupling between the spins. For simplicity, we assume that N is even. For completeness, we always mention also known results for fully separable states [5, 7]. We have the following theorems:

Theorem 1. *Let ϱ be an N qubit state of a system described by the Heisenberg Hamiltonian in equation (3). If*

$$\langle H_H \rangle < -JN =: C_{R2} \tag{4}$$

there exists two neighbouring qubits i and $i+1$ in the chain such that the reduced state $\varrho_{i,i+1}$ is a two-qubit entangled state. Furthermore, if

$$\langle H_H \rangle < -\frac{1+\sqrt{5}}{2} JN \approx -1.618 JN =: C_{R3} \tag{5}$$

then there exist three neighbouring qubits $i, i+1$ and $i+2$ such that the reduced state $\varrho_{i,i+1,i+2}$ of these qubits is genuine tripartite entangled.

Proof. It suffices to prove these bounds for pure states. To prove first equation (5), we write $H_H/J = W_{123} + W_{345} + W_{567} + \cdots$ with $W_{ijk} = \sigma_x^{(i)}\sigma_x^{(j)} + \sigma_y^{(i)}\sigma_y^{(j)} + \sigma_z^{(i)}\sigma_z^{(j)} + \sigma_x^{(j)}\sigma_x^{(k)} + \sigma_y^{(j)}\sigma_y^{(k)} + \sigma_z^{(j)}\sigma_z^{(k)}$. The idea is to view W_{ijk} as entanglement witnesses, detecting genuine tripartite

[3] This follows from the fact that there are three-qubit states which are separable for each fixed partition but not fully separable [10]. These states contain two-party entanglement, however, any reduced two-qubit state is separable.

entanglement on the qubits i, j and k. Let us denote $x_i = \langle\sigma_x^{(i)}\rangle$, $x_i x_j = \langle\sigma_x^{(i)}\sigma_x^{(j)}\rangle$ etc and consider a biseparable state $|\psi\rangle = |\phi_i\rangle|\phi_{jk}\rangle$. We have $|\langle W_{ijk}\rangle| = |x_i \cdot x_j + y_i \cdot y_j + z_i \cdot z_j + x_j x_k + y_j y_k + z_j z_k| \leqslant \sqrt{x_j^2 + y_j^2 + z_j^2} + |x_j x_k + y_j y_k + z_j z_k|$. For the two-qubit state $|\phi_{jk}\rangle$, the first term is the purity of the reduced state and invariant under local unitaries, while the second is maximal, when the 3×3 correlation matrix $\lambda_{\mu\nu} = \mu_j \nu_k$ for $\mu, \nu = x, y, z$ is diagonal. Expressing a general state in this form (see equation (14) in [11]) leads to $|\langle W_{ijk}\rangle| \leqslant 1 + \sqrt{5}$. Since there are $N/2$ of the W_{ijk}, equation (5) implies that one of the reduced three-qubit states cannot be biseparable. Finally, equation (4) can be proved similarly, using $|x_i \cdot x_j + y_i \cdot y_j + z_i \cdot z_j| \leqslant 1$ for separable two-qubit states [5, 7]. □

Theorem 2. *Let ϱ be an N qubit state as in theorem 1. If ϱ is one-producible, then*

$$\langle H_H\rangle \geqslant -JN =: C_{C2} \tag{6}$$

holds, while for two-producible states

$$\langle H_H\rangle \geqslant -\frac{3}{2}JN =: C_{C3} \tag{7}$$

holds. Thus, if $\langle H_H\rangle < C_{C3}$ the state contains genuine tripartite entanglement.

Proof. The bound equation (6) follows from equation (4) and has already been derived in [5, 7]. Let us first consider a two-producible pure state, where neighbouring spins are allowed to be entangled. i.e., we have a state of the type $|\psi\rangle = |\phi_{12}\rangle|\phi_{34}\rangle \cdots |\phi_{N-1,N}\rangle$, where the state $|\phi_{12}\rangle$ is a state of the qubits 1 and 2, etc. Let us define for the state $|\psi\rangle$ two vectors with $6N$ real components each, via

$$\begin{aligned} \vec{v}_1 &:= ([1], [1:2], [2], [\mathbb{1}], [5], [5:6], [6], \ldots, [\mathbb{1}]), \\ \vec{v}_2 &:= ([N], [\mathbb{1}], [3], [3:4], [4], [\mathbb{1}], [7], \ldots, [N-1:N]), \end{aligned} \tag{8}$$

where the symbols $[\cdots]$ stand always for three entries, namely $[i] = x_i, y_i, z_i$, $[i:j] = x_i x_j, y_i y_j, z_i z_j$ and $[\mathbb{1}] = 1, 1, 1$. Since the $\vec{v}_j$ have $6N$ entries, in $\vec{v}_1$ (and similarly for $\vec{v}_2$) some coefficients (like x_1) may appear twice, namely iff $N \in \mathbb{Z}$. It is straightforward to see that for the given state $2 \cdot \langle H_H\rangle = J\vec{v}_1 \cdot \vec{v}_2$ holds. Now, we need $\chi = x_i^2 + y_i^2 + z_i^2 + (x_i x_{i+1})^2 + (y_i y_{i+1})^2 + (z_i z_{i+1})^2 + x_{i+1}^2 + y_{i+1}^2 + z_{i+1}^2 \leqslant \chi_{\max} = 3$, which holds for any two-qubit state ϱ, due to $(1+\chi)/4 \leqslant Tr(\varrho^2) \leqslant 1$. Based on that, we have $\|\vec{v}_1\|^2 \leqslant (N/2)(\chi_{\max} + 3) = 3N$. The same bound holds for $\|\vec{v}_2\|^2$. So, due to the Cauchy–Schwarz inequality we have $|\langle H_H\rangle| \leqslant 1/2J\|\vec{v}_1\| \cdot \|\vec{v}_2\| \leqslant 3/2JN$ which proves the claim.

Now, we have to consider arbitrary two-producible states. A general two-producible pure state is always a tensor product of two-qubit states $|\phi_{ij}\rangle$ of the qubits i and j and single-qubit states $|\phi_k\rangle$. If for one of the two-qubit states $|\phi_{ij}\rangle$ the qubits i and j are not neighbouring, we can replace it by two one-qubit reduced states $\varrho_i \otimes \varrho_j$, since in this case, the Hamiltonian is only sensitive to the reduced states. Furthermore, if two neighbouring single-qubit states $|\phi_i\rangle \otimes |\phi_j\rangle$, appear, we replace them by $|\phi_{ij}\rangle$, since $|\phi_{ij}\rangle$ is allowed to be separable.

Thus, it suffices to prove the bound for a state where entanglement is only present between neighbouring qubits and the single-qubit states are isolated in the sense that if $|\phi_k\rangle$ appears, then $|\phi_{k-1}\rangle$ and $|\phi_{k+1}\rangle$ do not appear, e.g. $|\psi\rangle = |\phi_{12}\rangle|\phi_3\rangle|\phi_{45}\rangle|\phi_{67}\rangle|\phi_8\rangle$. Let M be the (even) number of isolated qubits. We can associate to any isolated qubit one neighbouring two-qubit state (on the left or right) to form a three-party group. There is an ambiguity in doing that, and we can choose the three-qubit groups in such a way that the number of two-qubit groups between the three-qubit groups is always even. Then, the state can be viewed as a sequence $g_1, g_2, g_3, g_4 \ldots$ of M three-party groups and $(N-3M)/2$ two party groups, all in all there are $(N-M)/2$ groups. We can double this sequence by setting $g_{(N-M)/2+k} = g_k$ etc. In the example, this grouping can be (12), (345), (678), (12), (345), (678). Now, we define the vectors $\vec{v}_1$ and $\vec{v}_2$ as follows: $\vec{v}_1$ collects terms from $g_1, g_3, g_5 \ldots$ while $\vec{v}_2$ consists of terms from $g_2, g_4, \ldots$ In detail, we have

$$\begin{aligned} \vec{v}_1 &= ([g_1], [g_1|g_2], [\mathbb{1}], [g_2|g_3], [g_3], [g_3|g_4], [\mathbb{1}], \ldots), \\ \vec{v}_2 &= ([\mathbb{1}], [g_1|g_2], [g_2], [g_2|g_3], [\mathbb{1}], [g_3|g_4], [g_4], \ldots), \end{aligned} \tag{9}$$

where $[g_1]$ denotes the three terms $x_i x_j$, $y_i y_j$ and $z_i z_j$ if g_1 is a two-qubit group and corresponds to the four terms $x_i x_j$, $y_i y_j$, $z_i z_j$ and $(x_j x_k + y_j y_k + z_j z_k)$ when g_1 is a three-qubit group with the isolated qubit k. $[g_1|g_2]$ etc denotes the coupling terms between the groupings g_1 and g_2, in the Hamiltonian, e.g. x_i, y_i, z_i for v_1 and $x_{i+1}, y_{i+1}, z_{i+1}$ for v_2 or vice versa. The symbol $[\mathbb{1}]$ denotes a sequence of three or four times '1', depending on whether in the other vector there is a two- or a three-qubit group.

Again, we have $2 \cdot \langle H \rangle = J \vec{v}_1 \cdot \vec{v}_2$ and it remains to bound $\|\vec{v}_i\|^2$. Firstly, note that by construction $\vec{v}_1$ and $\vec{v}_2$ contain the same number of two- and three-qubit groups, namely $(N-3M)/2$ two-qubit groups and M three-qubit groups each. Also, both contain $3(N-3M)/2 + 4M$ times the '1'. Then, note that for states of the type $|\psi\rangle = |\phi_{ij}\rangle|\phi_k\rangle$ the bound $x_i^2 + y_i^2 + z_i^2 + (x_i x_j)^2 + (y_i y_j)^2 + (z_i z_j)^2 + (x_j \cdot x_k + y_j \cdot y_k + z_j \cdot z_k)^2 + x_k^2 + y_k^2 + z_k^2 \leqslant 5$ is valid. So we have $\|\vec{v}_i\|^2 \leqslant (4+5)M + (3+3)(N-3M)/2 = 3N$, which proves the claim. □

To start the discussion, first note that the bounds in equations (6, 7) are sharp. Equation (7) is saturated for the singlet chain $|\psi\rangle = |\psi^-\rangle|\psi^-\rangle \cdots |\psi^-\rangle$ where $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.

In general, any energy bound $\langle H_H \rangle \geqslant C_X$ corresponds to a certain temperature T_X. Below this temperature the state has with certainty some degree of entanglement. Corresponding to the equations (4)–(7) there are thus the temperatures T_{R2}, T_{R3}, T_{C2} and T_{C3} below which either reduced states of two or three parties are entangled or the total state contains two- or three-party entanglement. Obviously, $T_{R2} = T_{C2} > T_{C3} > T_{R3}$ has to hold here.

Let us estimate these temperatures. For small N one can solve the Heisenberg model by diagonalizing H_H numerically[4]. Then, the threshold temperatures can directly be computed. Results are shown in figure 1. As expected, the values for $T_{C2} = T_{R2}$ coincide with the ones in [7]. The given values for T_{C3} and T_{R3} show that in the Heisenberg chain of ten spins at $kT \approx J$ multipartite entanglement plays a role, namely at least one reduced state is a genuine tripartite entangled and the total state contains tripartite entanglement.

In the thermodynamic limit $N \gg 1$ the ground-state energy of the Heisenberg model is known to be $E_0/N = -J(4\ln 2 - 1) \approx -1.773\,J$ [13]. Thus three qubits can be found such that their reduced state is genuine tripartite entangled. One can infer from numerical calculations [14] that the threshold temperatures are determined by $kT_{C3} \approx 1.61 J$ and $kT_{R3} \approx 1.23\,J$ in agreement with the values of the ten-qubit Heisenberg chain.

[4] Alternatively, one may also use the Bethe ansatz to derive the thermodynamic properties [12].

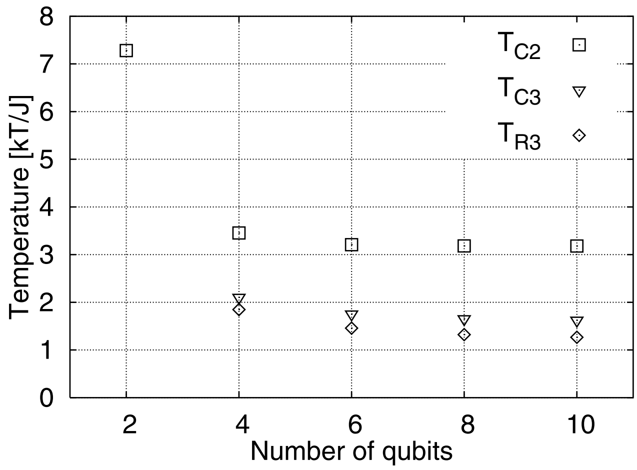

Figure 1. Threshold temperatures T_{R3}, T_{C2} and T_{C3} for small Heisenberg chains up to ten qubits. See text for details.

Finally, note that our results also shed light on the characterization of the ground state itself. For instance, they show that the ground state cannot be a GHZ state since for that state the reduced three-qubit states are separable. Finally, note that when a state is translationally invariant, equation (5) guarantees that *all* reduced three-qubit states are genuine tripartite entangled.

4. The XY model

Let us now investigate the isotropic XY model. The Hamiltonian of this model is given by

$$H_{XY} = J \sum_i \sigma_x^{(i)} \sigma_x^{(i+1)} + \sigma_y^{(i)} \sigma_y^{(i+1)}. \tag{10}$$

Again, we assume periodic boundary conditions and an even number of spins. For this model we have:

Theorem 3. *Let ϱ be an N qubit state whose dynamics is governed by the Hamiltonian in equation (10). If ϱ is one-producible, then*

$$\langle H_{XY} \rangle \geqslant -JN \tag{11}$$

holds. If $\langle H_{XY} \rangle < -JN$ this implies that there are two neighbouring qubits such that their reduced state is entangled. For two-producible states

$$\langle H_{XY} \rangle \geqslant -\frac{9}{8} JN \tag{12}$$

holds. If $\langle H_{XY} \rangle < -9/8JN$ the state contains thus tripartite entanglement and if

$$\langle H_{XY} \rangle < -\frac{1+\sqrt{2}}{2} JN \approx -1.207JN, \tag{13}$$

then there are three neighbouring qubits such that their reduced state is genuine tripartite entangled.

Proof. The proofs for the XY model are similar to the ones for the Heisenberg model, so we can make it short. equation (11) can be proved in the same manner as in the Heisenberg model [5]. Now, let us first show the bound (12) for a state of the type $|\psi\rangle = |\phi_{12}\rangle|\phi_{34}\rangle \cdots |\phi_{N-1,N}\rangle$. Again, we define two vectors $\vec{v}_1$ and $\vec{v}_2$, now with $4N$ entries via $\vec{v}_1 := ([1], [1:2], [2], [\mathbb{1}], [5], [5:6], [6], [\mathbb{1}], \ldots, [\mathbb{1}])$ and $\vec{v}_2 := ([N], [\mathbb{1}], [3], [3:4], [4], [\mathbb{1}], [7], [7:8], \ldots, [N-1:N])$, where now $[i] = x_i, y_i$ and $[i:j] = \alpha\beta_{1,2}, \alpha\beta_{1,2}$ and $[\mathbb{1}] = \alpha^{-1}, \alpha^{-1}$ with $\alpha = \sqrt{4/3}$ and $\beta_{i,j} = (x_i x_j + y_i y_j)/2$. Again, we have $2\langle H_{XY}\rangle = J\vec{v}_1 \cdot \vec{v}_2$ and due to the fact that $x_i^2 + y_i^2 + 2\alpha^2\beta_{i,j}^2 + y_j^2 + x_j^2 \leqslant \xi_{\max} = 3$, we have that $\|\vec{v}_1\|^2 \leqslant N/2(\xi_{\max} + 2/\alpha^2) = 9/4N$, which proves the claim. General two-producible states can be treated as shown in the proof of theorem 2, now for states of the type $|\psi_{ij}\rangle \otimes |\psi_k\rangle$ the bound $x_i^2 + y_i^2 + 2\alpha^2\beta_{i,j}^2 + \alpha^2(x_j \cdot x_k + y_j \cdot y_k)^2 + y_k^2 + x_k^2 \leqslant x_i^2 + y_i^2 + 2\alpha^2\beta_{i,j}^2 + \alpha^2(x_j^2 + y_j^2) + 1 \leqslant 9/2$ must be used. Finally, equation (11) can be proved as equation (5), using $\sqrt{x_j^2 + y_j^2} + |x_j x_k + y_j y_k| \leqslant 1 + \sqrt{2}$. □

Firstly, note that again the bounds in equations (11), (12) are sharp. In equation (12) equality holds for the state $|\psi\rangle = |\phi_{12}\rangle|\phi_{34}\rangle \cdots |\phi_{N-1,N}\rangle$ where $|\phi_{i,i+1}\rangle\langle\phi_{i,i+1}| = 1/4 \cdot (\mathbb{1} \otimes \mathbb{1} - \sigma_x \otimes \sigma_x - 1/2 \cdot [\sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z] - \sqrt{3}/2 \cdot [\mathbb{1} \otimes \sigma_x - \sigma_x \otimes \mathbb{1}])$ for all odd i. This corresponds to a chain of non-maximally entangled two-qubit states. This also motivates the definition of α and $\beta_{i,j}$ in the proof of equation (12). α and $\beta_{i,j}$ were chosen such that for the state $|\psi\rangle$ we have $\vec{v}_1 = \vec{v}_2$, thus the Cauchy–Schwarz inequality is sharp.

In the thermodynamic limit the XY model is exactly solvable [15]. The ground-state energy is $E_0/N = -4J/\pi \approx -1.273\,J$. Thus, the ground state contains tripartite entanglement and fulfils the condition of equation (13). Furthermore, the results of [15] imply that if $kT < 0.977\,J$, the mixed state contains tripartite entanglement and if $kT < 0.668\,J$, some reduced state is genuine tripartite entangled.

5. Conclusion

In conclusion, we have shown for two important spin models that the internal energy can be used as a signature for the presence of multipartite entanglement in these models. Based on this, we computed threshold temperatures below which any realistic description of the system cannot neglect the effects of multipartite entanglement. Our results may stimulate the research on entanglement in phase transitions, since they suggest the use of multipartite entanglement measures as a tool for the investigation of phase transitions in these regimes [16].

A natural continuation of the present work is the extension of the theorems presented here to other spin systems, e.g. higher dimensional or frustrated systems. Furthermore, it is very tempting to analyse the results of performed experiments whether they can be interpreted as giving evidence for multipartite entanglement, as has similarly been done for bipartite entanglement[5]. Here, we leave this as an open problem.

[5] See [17]. Note, however, that the system investigated there can be described well by a two-producible state, thus it is not suitable for the search for multipartite entanglement.

Acknowledgments

We thank J I Cirac, W Dür, J J García-Ripoll, A Miyake, M Plenio and especially T Xiang for discussions. This work has been supported in part by the EU (MEIF-CT-2003-500183, IST-2001-38877, OLAQUI, SCALA), the FWF and the DFG.

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