

# Entanglement detection based on interference and particle counting

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A sufficient condition for entanglement in two-mode continuous systems is constructed with the interference visibility and the uncertainty of the total particle number. The observables to be measured (particle numbers and particle number variances) are relatively easily accessible experimentally. The method may be used to detect entanglement in light fields or in Bose-Einstein condensates. In contrast to the standard approach based on entanglement witnesses, the condition is expressed in terms of an inequality which is nonlinear in expectation values. The condition is constructed using uncertainty relations with the particle number and the destruction operators.

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## I. INTRODUCTION

In spite of considerable efforts, the separability of general mixed quantum states is still an open problem, even if the whole density matrix is known. The positivity of the partial transpose [1] of the density matrix is a necessary condition for separability; however, it is a sufficient condition only for the  $2 \times 2$  (two qubits) and  $2 \times 3$  dimensional cases and for two-mode Gaussian states in continuous variable systems. For higher dimensions there exist entangled states [2] with positive partial transpose and the separability problem is not fully solved. For bipartite low-rank density matrices an operational criterion for separability is presented in Ref. [3]. A necessary and sufficient condition for entanglement of all bipartite Gaussian continuous states is also known [4]. Entanglement can also be detected by several other methods, e.g., through witness operators whose expectation value is negative only for (some) entangled states [5].

In an experiment the density matrix is usually not known, only partial information is available on the quantum state. One can typically measure a few observables and still would like to detect some of the entangled states [6]. Thus to find a criterion for entanglement with easily measurable observables is crucial for entanglement detection. There are only few such criteria in the literature [7–11] for detecting entanglement in complicated situations such as in many-particle or continuous variable systems. The method described in Ref. [7] detects entanglement among cold atoms having two internal degrees of freedom based on inequalities with the total angular momentum components. Reference [8] derives a criterion for the entanglement between two pairs of bosonic modes in terms of the total angular momentum and the particle number. This criterion detects entangled states that are close to singlet states of two large spins.

A criterion for detecting entanglement between two modes is given in Refs. [9,10]. Reference [9] presents a scenario where one just has to measure the second moments of  $x$  and  $p$  for both systems. For example, if the inequality

$$[\Delta(x_A + x_B)]^2 + [\Delta(p_A - p_B)]^2 < 2 \quad (1)$$

is fulfilled, then the state is entangled. Here  $(\Delta X)^2 = \langle X^2 \rangle$

$-\langle X \rangle^2$ , and  $x_k$  and  $p_k$  are canonical operators satisfying  $[x_k, p_k] = i$ . Note that the left-hand side of Eq. (1) is quadratic in expectation values. However, by setting  $\langle x_k \rangle$  and  $\langle p_k \rangle$  to zero by single-party unitary operations, only terms linear in expectation values remain and the criterion is equivalent to an entanglement witness.

A generalization of criterion (1) detects all entangled two-mode Gaussian states [9,10]. However, in many experimental situations non-Gaussian states are prepared. For example, if one has  $N$  photons and sends them through a beam splitter or if one has  $N$  atoms in some internal state and applies an appropriate laser pulse, the state will be

$$|\Psi\rangle = \frac{1}{\sqrt{2^N N!}} (a^\dagger + b^\dagger)^N |0,0\rangle = \frac{1}{\sqrt{2^N}} \sum_{n=0}^N \sqrt{\binom{N}{n}} |n, N-n\rangle. \quad (2)$$

Here  $a$  and  $b$  are annihilation operators which are defined according to  $x_A = (a + a^\dagger)/\sqrt{2}$ . This entangled state is not detected by the previous criterion as it will be demonstrated later.

In this paper, we will present a criterion which: (i) requires measuring quantities which are easily accessible experimentally and (ii) detects entangled states in the vicinity of state (2). The paper is organized as follows. In Sec. II the entanglement criterion is derived. In Sec. III it is discussed what states are detected by this criterion and issues concerning its possible experimental applications are also considered. A summary is given in Sec. IV.

## II. ENTANGLEMENT CRITERION

In this section we will show that for all separable states, i.e., states that can be written as

$$\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B, \quad (3)$$

the following expression involving the variances of the total particle number  $N := a^\dagger a + b^\dagger b$  and of the operator  $(a - b)$  is bounded from below as

$$\{(\Delta_\rho N)^2 + 1\}\{[\Delta_\rho(a-b)]^2 + 1\} \geq \frac{\langle N \rangle_\rho}{4} + \frac{1}{8}, \quad (4)$$

where  $(\Delta_\rho A)^2 := \langle A^\dagger A \rangle_\rho - |\langle A \rangle_\rho|^2$  (note that  $A$  need not be Hermitian).

The physical motivation for this criterion comes from the observation, made in the context of Bose-Einstein condensates in Ref. [12] (also explained in Sec. III), that it is not possible to have a fixed total particle number—corresponding to  $(\Delta_\rho N)^2 = 0$ —and perfect interference—corresponding to  $[\Delta_\rho(a-b)]^2 = 0$ —at the same time, unless the system under consideration is in a highly nonclassical, i.e., entangled, state.

In Sec. II A a simple separability criterion will be proved. In Sec. II B this criterion will be generalized. Technical details are in the Appendix. Section II C proves our main result, Eq. (4).

### A. Simple criterion

In this section a simple separability criterion will be derived based on uncertainty relations for the two subsystems. In order to understand the connection between the uncertainty relations and the necessary condition for separability, it is instructive first to review how the condition (1) was obtained starting from the single-subsystem uncertainty relations [9,11],

$$(\Delta_\rho x_{A/B})^2 + (\Delta_\rho p_{A/B})^2 \geq 1. \quad (5)$$

For a separable state of the form (3) the sum of uncertainties of the EPR-type operators  $x_A + x_B$  and  $p_A - p_B$  can be written as

$$\begin{aligned} & [\Delta_\rho(x_A + x_B)]^2 + [\Delta_\rho(p_A - p_B)]^2 \\ &= \sum_k p_k \{ (\Delta_{\rho_k^A} x_A)^2 + (\Delta_{\rho_k^A} p_A)^2 + (\Delta_{\rho_k^B} x_B)^2 + (\Delta_{\rho_k^B} p_B)^2 \} \\ &+ \sum_k p_k \{ \langle (x_A + x_B) \rangle_{\rho_k^A \otimes \rho_k^B}^2 - \langle (x_A + x_B) \rangle_\rho^2 \\ &+ \langle (p_A - p_B) \rangle_{\rho_k^A \otimes \rho_k^B}^2 - \langle (p_A - p_B) \rangle_\rho^2 \} \geq 2. \end{aligned} \quad (6)$$

In the equality it was exploited that for a product state the uncertainty of an EPR-type operator splits into the sum of the corresponding single system uncertainties as  $[\Delta_{\rho_k^A \otimes \rho_k^B}(x_A + x_B)]^2 = (\Delta_{\rho_k^A} x_A)^2 + (\Delta_{\rho_k^B} x_B)^2$ . Based on the uncertainty relations (5) for the individual subsystems,  $A$  and  $B$ , the first sum in Eq. (6) is bounded from below by 2. Since the second sum is non-negative, the left-hand side of Eq. (6) is also bounded by 2 which finishes the proof of Eq. (1). Thus for separable states the sum of the variances of  $x_A + x_B$  and  $p_A - p_B$  has the same lower bound as the sum of the corresponding single system uncertainties. Note that this simple relationship holds because the right-hand side of the uncertainty relation (5) is a *constant*. On the other hand, this

bound is not valid for nonseparable states. In this case the sum of the two uncertainties can even be zero [9] since  $x_A + x_B$  and  $p_A - p_B$  commute.

After reviewing the previous example with the EPR operators, we will prove that all separable states fulfill

$$\Sigma_\rho := (\Delta_\rho N)^2 + [\Delta_\rho(a-b)]^2 \geq f(\langle N \rangle_\rho), \quad (7)$$

where

$$f(N) = \sqrt{N + \frac{3}{4}} + \frac{\sqrt{3}}{2} - 2. \quad (8)$$

This will be necessary later to obtain our main result, Eq. (4). Here  $(\Delta_\rho N)^2$  is the variance of the total particle number in the two-mode system (i.e., in a two-mode electromagnetic field or Bose-Einstein condensate in a double-well potential), while  $[\Delta_\rho(a-b)]^2$  is related to the variance of the phase difference between the two modes.

The proof of Eq. (7) is the following. For a separable state of the form (3), the sums of the two uncertainties in Eq. (7) can be written as  $\Sigma_\rho = \Sigma_{\rho,0} + \Sigma_{\rho,1}$ , where

$$\begin{aligned} \Sigma_{\rho,0} &= \sum_k p_k \{ (\Delta_{\rho_k^A} N_A)^2 + (\Delta_{\rho_k^A} a)^2 + (\Delta_{\rho_k^B} N_B)^2 \\ &+ (\Delta_{\rho_k^B} b)^2 + N_k^2 - \langle N \rangle_\rho^2 \}, \end{aligned} \quad (9)$$

$$\Sigma_{\rho,1} = \sum_k p_k \{ \langle (a-b) \rangle_{\rho_k}^2 - |\langle (a-b) \rangle_\rho|^2 \}. \quad (10)$$

Here  $N_A := a^\dagger a$ ,  $N_B := b^\dagger b$ , and  $N_k := \langle a^\dagger a + b^\dagger b \rangle_{\rho_k}$ . Using the Cauchy-Schwarz inequality, one can show that  $\Sigma_{\rho,1} \geq 0$ , and thus  $\Sigma_\rho \geq \Sigma_{\rho,0}$ .

Next, we will need the following uncertainty relation proved in part 1 and 2 of the Appendix

$$R_\rho := (\Delta_\rho N_A)^2 + (\Delta_\rho a)^2 \geq L(\langle N_A \rangle_\rho), \quad (11)$$

where

$$L(N) = \sqrt{N + \frac{3}{4}} - 1. \quad (12)$$

Obviously, the same inequality is true for subsystem  $B$ . Inequality (11) is an alternative of the number-phase uncertainty without the problem of defining an appropriate phase operator and the difficulties due to the  $2\pi$  periodic nature of the phase [13].

In our case the bound in the uncertainty relation (11) is not a constant but a *function* of an operator expectation value. Thus the method presented for the EPR-type operators cannot be used, and careful analysis of the different properties of the function  $L(N)$  must be done [14]. Using inequality (11) and the fact that bound (12) fulfills  $L(N_1) + L(N_2) \geq L(N_1 + N_2) + L(0)$  we obtain

$$\Sigma_\rho \geq \sum_k p_k \{ L(N_k) + L(0) + N_k^2 - \langle N \rangle_\rho^2 \}. \quad (13)$$

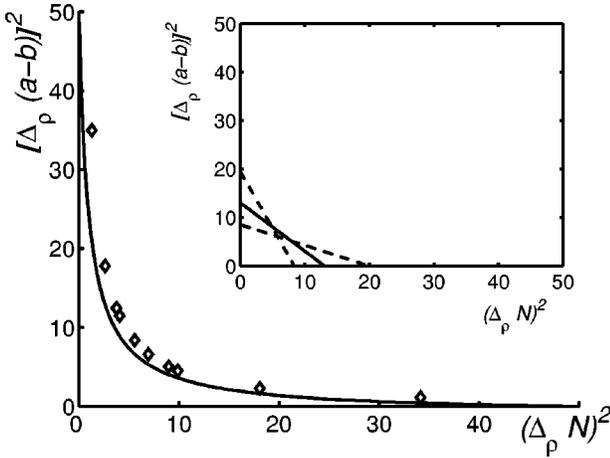


FIG. 1. Numerical test of the inequality (4) for the two-mode separability problem for  $N=200$ . (Solid) Boundary of the region defined by Eq. (4). All states below this line are entangled. (Diamonds) Points corresponding to separable states found numerically. Inset: (solid) boundary of the region defined by Eq. (7); (dashed) boundary of the region defined by Eq. (14) for  $w=0.3$  and  $0.7$ .

Now using the fact that  $L(N_k) + N_k^2$  is a concave function of  $N_k$ , we obtain  $\sum_{\rho} \geq L(N) + L(0)$  which proves Eq. (7).

Condition (7) corresponds to a line on the  $(\Delta_{\rho}N)^2 - [\Delta_{\rho}(a-b)]^2$  plane (solid line in the inset of Fig. 1). All separable states belong to points above this line and fulfill Eq. (7). Points below this line correspond to entangled states only.

### B. Generalization

We would like to find more entangled states in the  $(\Delta_{\rho}N)^2 - [\Delta_{\rho}(a-b)]^2$  plane. In order to do that we generalize Eq. (7) as

$$\sum_{\rho,w} := w(\Delta_{\rho}N)^2 + (1-w)[\Delta_{\rho}(a-b)]^2 \geq f_w(\langle N \rangle_{\rho}), \quad (14)$$

where  $0 < w < 1$  determines the relative weights of the two terms and  $f_w(N)$  is defined at the end of this section. Inequality (14) corresponds a region above a line with slope  $w/1-w$  (dashed lines in the inset of Fig. 1). These lines are the tangentials of the curve enclosing all separable states. All the points below this curve correspond to entangled states.

In order to obtain the lower bound  $f_w(\langle N \rangle_{\rho})$ , we have to follow a procedure similar to what was presented in the preceding section. For a separable state one obtains

$$\begin{aligned} \sum_{\rho,w} \geq \sum_k p_k \{ & w(\Delta_{\rho_k^A} N_A)^2 + (1-w)(\Delta_{\rho_k^A} a)^2 + w(\Delta_{\rho_k^B} N_B)^2 \\ & + (1-w)(\Delta_{\rho_k^B} b)^2 + wN_k^2 - w\langle N \rangle_{\rho_k}^2 \}. \end{aligned} \quad (15)$$

In Appendix A 3 we prove the following uncertainty relation:

$$R_{\rho,w} := w(\Delta_{\rho} N_A)^2 + (1-w)(\Delta_{\rho} a)^2 \geq L_w(\langle N_A \rangle_{\rho}), \quad (16)$$

where

$$L_w(N) = \begin{cases} \sqrt{w(1-w)\left(N + \frac{1}{4}\right) + \frac{w}{4} - \frac{1}{2}} & \text{if } N \geq N_L \\ (N - N_L)w(1-w) & \text{if } N < N_L. \end{cases} \quad (17)$$

Here  $N_L = (1-w)/4w$ . Inequality (16) is the generalization of Eq. (11) for unequal weights for the two variances. For  $N < N_L$  the function  $L_w(N)$  is linear and the slope is determined in such a way that there is not an abrupt change in the derivative of  $L_w(N)$  at  $N = N_L$ .

In order to get a lower bound for  $\sum_{\rho,w}$  using the uncertainty relation (16), one has to follow similar steps as in Sec. II A. Using the facts that  $L_w(N_1) + L_w(N_2) \geq L_w(N_1 + N_2) + L_w(0)$  is fulfilled and  $L_w(N) + wN^2$  is a concave function of  $N$ , the lower bound for  $\sum_{\rho,w}$  is obtained as  $f_w(N) = L_w(N) + L_w(0)$ .

### C. Proof of main result

In this section Eq. (4) will be obtained by determining the curve which has the lines corresponding to different  $w$ 's as its tangentials. The tangentials of a hyperbola  $(y+c_0)/(x+c_0) = C/(x+c_0)$  are given by  $wx + (1-w)y = 2\sqrt{w(1-w)C} - c_0$ . One can reformulate Eq. (14) by replacing the right-hand side by a slightly weaker lower bound which fits this form,

$$\tilde{f}_w(N) = \sqrt{w(1-w)\left(N + \frac{1}{2}\right)} - 1. \quad (18)$$

Hence the equation for a hyperbola on the  $(\Delta_{\rho}N)^2 - [\Delta_{\rho}(a-b)]^2$  plane can be obtained (solid curve in Fig. 1). Inequality (4) corresponds to points above this hyperbola. Any state which violates this inequality is necessarily entangled.

### III. DISCUSSION

First, the tightness of the necessary condition for separability (4) should be verified. Numerical checks show that it is quite strong (see Fig. 1). The diamonds indicate product states of the form  $|0\rangle \otimes |\Psi\rangle$  found numerically. The state in the origin of Fig. 1, giving zero for both variances in Eq. (4), is state (2) as can be shown as follows. Eigenstates of  $\hat{N}$  with  $N$  particles have the form  $|\Psi\rangle = \sum c_n |n, N-n\rangle$ . The state  $(a-b)|\Psi\rangle$  has  $N-1$  particles, thus  $(a-b)|\Psi\rangle = \lambda|\Psi\rangle$  is possible only if the eigenvalue  $\lambda=0$ . A state for which  $(a-b)|\Psi\rangle=0$  has to fulfill  $c_{n+1}\sqrt{n+1} = c_n\sqrt{N-n}$ . This determines the state uniquely as Eq. (2).

Our method detects entangled states in the proximity of state (2) on the  $(\Delta N)^2 - [\Delta(a-b)]^2$  plane as shown in Fig. 1. (In this section we will omit the  $\rho$  index.) Other interesting states on the  $(\Delta N)^2 - [\Delta(a-b)]^2$  plane: A separable state having  $(\Delta N)^2=0$  is the convex combination of products of Fock states  $|n_k\rangle|N-n_k\rangle$ . For these  $[\Delta(a-b)]^2=N$ . Separable states with perfect destructive interference between the modes having  $[\Delta(a-b)]^2=0$  are the convex combination of products of coherent states of the form  $|\alpha_k\rangle|\alpha_k+c\rangle$ , where  $c$  is a constant common for all product subensembles. For these states  $(\Delta N)^2 \geq N$  [12].

According to our criterion, to detect entanglement in an experiment, the variances of  $N$  and  $(a-b)$  should be measured. A simpler scenario is to measure the variance of  $N$  and  $\langle (a^\dagger - b^\dagger)(a-b) \rangle$ . Since  $[\Delta(a-b)]^2 \leq \langle (a^\dagger - b^\dagger)(a-b) \rangle$  inequality (7) can be used for this case after replacing  $(\Delta(a-b))^2$  by  $\langle (a^\dagger - b^\dagger)(a-b) \rangle$ . The latter is just twice the particle number in the mode  $b' = (a-b)/\sqrt{2}$ .

The generation of state (2) is never perfect, thus the system is in a mixed state. Our method makes it possible to detect entanglement even in this case. If  $\langle b'^\dagger b' \rangle \approx (\Delta N)^2$  the maximum particle number variance for a state to be detected is  $(\Delta N)^2 \propto \sqrt{N}$ , which is much smaller than for coherent states. On the other hand, for perfect destructive interference when  $\langle b'^\dagger b' \rangle \approx 0$  the maximal variance is  $(\Delta N)^2 \propto N$ .

Equation (2) describes the quantum state of a Bose-Einstein condensate of atoms, if the  $a$  and  $b$  modes correspond to the two halves of the condensate [12]. In this case  $(a^\dagger + b^\dagger)$  creates a particle in state  $|\Psi\rangle$  and Eq. (2) describes a product of single-particle states of the form  $|\Psi\rangle \otimes |\Psi\rangle \otimes \dots \otimes |\Psi\rangle$ . Although it is a product state from the point of view of the individual particles, in the  $|n, m\rangle$  basis it is clearly entangled. In order to detect entanglement, one needs to measure the variance of the total particle number and the particle number in one of the new modes after the two halves of the condensates interfere [15,16].

The condensate can be “split” into two modes, realizing state (2), and then reunited for detection in a Mach-Zehnder type interferometer [16]. The state (2) can also be obtained in a Bose-Einstein condensate of two-level atoms, by preparing the atoms in the same internal state and then applying a  $\pi/2$  laser pulse. The modes can then be spatially separated with a state-dependent potential [17].

Finally, the state (2) can be prepared with a 50/50 beam splitter and a laser pulse corresponding to a state with low photon number variance. For obtaining  $(\Delta N)^2$  and  $\langle b'^\dagger b' \rangle$ , a second beam splitter can be used, together with photon number measurements in the two modes. In order to detect entanglement, assuming perfect destructive interference at the second beam splitter, for the photon source  $(\Delta N)^2 \leq N/4 - 7/8$  is required. This can be obtained, for example, with a state with sub-Poissonian number statistics.

Besides experimental considerations, the advantage of our approach is the ability to detect states in the vicinity of the entangled state (2) which is not detected by the method based on the correlation matrix [9,10]. The correlation matrix  $\gamma$  contains the correlations of two pairs of conjugate single-party observables, which now we choose to be  $\{R_k\} = \{x_A, p_A, x_B, p_B\}$ . Here  $x_A = (a + a^\dagger)/\sqrt{2}$ ,  $p_A = (a - a^\dagger)/(\sqrt{2}i)$ , and  $x_B$  and  $p_B$  are defined similarly for the  $b$  mode. For the state (2) the correlation matrix  $\gamma_{kl} = \text{Tr}\{\rho(R_k - \langle R_k \rangle)(R_l - \langle R_l \rangle)\} + \text{Tr}\{\rho(R_l - \langle R_l \rangle)(R_k - \langle R_k \rangle)\}$  is obtained as

$$\gamma = \begin{pmatrix} N+1 & 0 & N & 0 \\ 0 & N+1 & 0 & N \\ N & 0 & N+1 & 0 \\ 0 & N & 0 & N+1 \end{pmatrix}. \quad (19)$$

The sufficient condition for inseparability is  $\tilde{\gamma} - iJ \neq 0$ , where  $\tilde{\gamma}$  is the correlation matrix corresponding to the partially transposed density matrix and  $J_{kl} = i[R_k, R_l]$ . Here  $\tilde{\gamma} - iJ \geq 0$  thus the state is not detected as entangled.

Moreover, with the simple method used for criterion (1) described in the Introduction, our criteria (4), (7), and (14) cannot be reduced to an entanglement witness. This is because they contain the variance of the particle number and  $\langle N \rangle$  cannot be set to zero by single-party unitary operations.

#### IV. CONCLUSIONS

In summary, a simple inequality for the expectation values of observables was proposed for entanglement detection. Since only the measurement of easily accessible quantities (particle numbers and particle number variances) are needed, this approach may be feasible for detecting entanglement experimentally in Bose-Einstein condensates or in a two-mode photon field.

Our method can be generalized for detecting other highly entangled states. First two operators must be identified which have the state as an eigenstate. Then a necessary condition for separability must be constructed with the variances of these operators. Such a highly entangled state is, for example, the  $|N, 0\rangle + |0, N\rangle$  Schrödinger cat state which is the eigenstate of  $N$  and  $(a^\dagger b)^N + (ab^\dagger)^N$ .

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#### APPENDIX: SINGLE-MODE UNCERTAINTY RELATION

##### 1. Analytic calculation

In this section we will prove Eq. (11). We will find a lower bound for the sum of the two variances  $(\Delta_\rho N_A)^2$  and  $(\Delta_\rho a)^2$  for any single-mode quantum state. This uncertainty relation is needed to find a lower bound for the sum of operator variances for two-mode separable states in Eq. (7).

The first term on the left-hand side of Eq. (11) is zero for number states. The second term is zero for coherent states.  $N_A$  and  $a$  have a common eigenvector: for the state  $|0\rangle$  the variances of both are zero. In order to find a nontrivial relation, the lower bound for the sum of the two variances must have at least one parameter. We choose this parameter to be  $\langle N_A \rangle_\rho$ . For  $\langle N_A \rangle_\rho > 0$  the operators  $N_A$  and  $a$  do not have common eigenvectors and the sum of the two variances are bounded from below.

The proof of Eq. (11) is based on finding two lower bounds for the left hand side of Eq. (11) and then combining them. Let us denote  $\langle a \rangle_\rho = \sqrt{\alpha \langle N_A \rangle_\rho} e^{i\phi}$ , where  $0 \leq \alpha \leq 1$ . The first lower bound is

$$R_\rho \geq (\Delta_\rho a)^2 = (1 - \alpha) \langle N_A \rangle_\rho =: B_1(N_A, \alpha). \quad (\text{A1})$$

The second bound [18] is obtained from

$$\begin{aligned} R_\rho &= (\Delta_\rho N_A)^2 + \frac{1}{2} [(\Delta_\rho x_A)^2 + (\Delta_\rho p_A)^2] - \frac{1}{2} \\ &\geq \sqrt{2(\Delta_\rho N_A)^2 [(\Delta_\rho x_A)^2 + (\Delta_\rho p_A)^2]} - \frac{1}{2}. \end{aligned} \quad (\text{A2})$$

Here for the inequality  $X^2 + Y^2 \geq 2XY$  was applied. Now, using the facts that  $(\Delta_\rho N_A)^2 (\Delta_\rho x_A)^2 \geq |\langle p_A \rangle_\rho|^2 / 4$  and  $(\Delta_\rho N_A)^2 (\Delta_\rho p_A)^2 \geq |\langle x_A \rangle_\rho|^2 / 4$  we obtain

$$\begin{aligned} R_\rho &\geq \sqrt{\frac{|\langle x_A \rangle_\rho|^2 + |\langle p_A \rangle_\rho|^2}{2}} - \frac{1}{2} \\ &= |\langle a \rangle| - \frac{1}{2} \\ &= \sqrt{\alpha \langle N_A \rangle_\rho} - \frac{1}{2} \\ &=: B_2(N_A, \alpha). \end{aligned} \quad (\text{A3})$$

From Eqs. (A1) and (A3) one can derive a higher lower bound for Eq. (11) by taking the maximum of these two bounds. It can be shown that

$$B(N, \alpha) := \max[B_1, B_2] = \begin{cases} B_1(N, \alpha) & \text{if } \alpha \geq \alpha_L, \\ B_2(N, \alpha) & \text{if } \alpha \leq \alpha_L, \end{cases} \quad (\text{A4})$$

where

$$\sqrt{\alpha_L} = \sqrt{1 + \frac{3}{4N}} - \frac{1}{2\sqrt{N}}. \quad (\text{A5})$$

Here  $\alpha_L$  is always non-negative, however, for small particle numbers  $N < 1/4$  it is larger than 1.

The lower bound for Eq. (11) will be constructed by minimizing  $B(N, \alpha)$  with respect to  $\alpha$ . After some algebra one obtains

$$\min_\alpha B(N, \alpha) = \begin{cases} \sqrt{N + \frac{3}{4}} - 1 & \text{if } N > \frac{1}{4}, \\ 0 & \text{if } N \leq \frac{1}{4}. \end{cases} \quad (\text{A6})$$

As stated in Sec. II A, in order to use this result in the two-mode separability problem the bound should fulfill two criteria (i)  $L(N) + N^2$  should be concave, (ii)  $L(N_1) + L(N_2) \geq L(N_1 + N_2) + L(0)$ . The bound Eq. (A6) does not fulfill (ii), thus a weaker bound satisfying this condition has to be chosen.  $L(N)$ , as defined in Eq. (12), is such a bound. It coincides with Eq. (A6) for  $N \geq 1/4$ , while for  $N < 1/4$  it is negative.

## 2. Numerical verification

In this section we prove by numerical calculations that Eq. (12) is a tight lower bound for Eq. (11). We will deter-

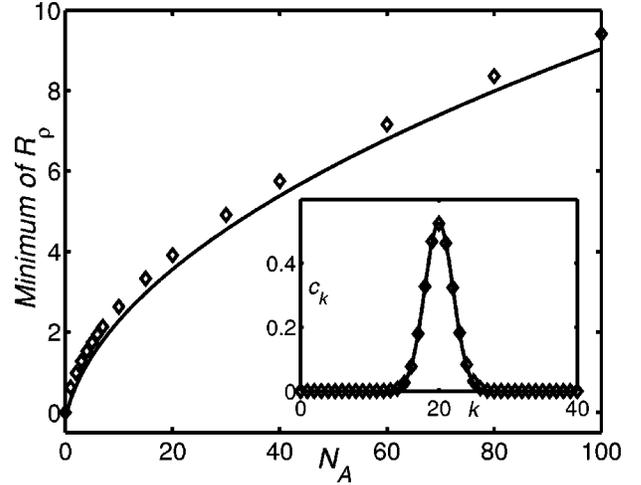


FIG. 2. (Diamonds) The minimum of  $R_\rho$  [for a definition see Eq. (11)] if the particle number is constrained to  $N_A$ . (Solid) The analytic lower bound  $L(N_A)$  defined in Eq. (12). The inset shows the wave function in the number basis (diamonds) corresponding to the minimum of uncertainties for  $N_A = 20$ . The results fit very well a Gaussian curve (solid).

mine the state vector minimizing the left-hand side of Eq. (11) and the corresponding minimum with the constraint  $\langle a^\dagger a \rangle = N_A$ .

The wave function is given in the number basis as

$$|\Psi\rangle = \sum_k c_k |k\rangle. \quad (\text{A7})$$

The left-hand side of Eq. (11) can be rewritten as

$$R_\rho = \left\{ \sum_k |c_k|^2 k^2 - N_A^2 \right\} + \left\{ N_A - \left| \sum_k c_k^* c_{k+1} \sqrt{k+1} \right|^2 \right\}. \quad (\text{A8})$$

Lagrange multipliers must be added in order to constrain the particle number to  $N_A$  and keep the norm one

$$\begin{aligned} g(\{c_m\}, \{c_m^*\}, \mu_1, \mu_2) \\ = R_\rho - \mu_1 \left( N_A - \sum_k |c_k|^2 k \right) - \mu_2 \left( 1 - \sum_k |c_k|^2 \right). \end{aligned} \quad (\text{A9})$$

When minimized, all the derivatives of Eq. (A9) must be zero. Moreover, since  $R_\rho(\{c_k\}) \geq R_\rho(\{|c_k|\})$  we can restrict our search for the minimum for real  $c_k$ 's. Hence one obtains

$$c_{n+1} = \left( \frac{n^2 + \mu_1 n + \mu_2}{A \sqrt{n+1}} \right) c_n - \left( \sqrt{\frac{n}{n+1}} \right) c_{n-1}, \quad (\text{A10})$$

where  $A = \langle a \rangle = \sum_k c_k c_{k+1} \sqrt{k+1}$  and the term with  $c_{n-1}$  is not present for  $n = 0$ . From  $A$ ,  $\mu_1$ , and  $\mu_2$  the unnormalized wave function can be constructed by setting  $c_0 = 1$ . As can be seen in Fig. 2,  $L(N_A)$  defined in Eq. (12) is very close to the minimum found numerically, thus it is a tight bound. The

wave function minimizing  $R_\rho$  is shown in the inset. In the number basis it fits very well a Gaussian curve even for small particle numbers.

### 3. Generalized single-mode uncertainty relation

In this section we will prove Eq. (16). For  $w=0$  the state minimizing  $R_{\rho,w}$  is a coherent state, for  $w=1$  it is a number state. For intermediate  $w$ 's the wave function giving the minimum interpolates between these two. The  $L_w(\langle N_A \rangle_\rho)$  bound can be obtained, after inserting  $w$  and  $(1-w)$  in the expression to be minimized, by following the same steps as in part 1 of the Appendix. The two bounds found will be

$$B_{1,w}(N_A, \alpha) = (1-w)(1-\alpha)N_A,$$

$$B_{2,w}(N_A, \alpha) = \sqrt{w(1-w)\alpha N_A} - \frac{1-w}{2}. \quad (\text{A11})$$

The maximum of these two,  $B_w(N, \alpha)$ , can be obtained knowing that  $B_{1,w}(N, \alpha) > B_{2,w}(N, \alpha)$  if  $\alpha > \alpha_L$ , where

$$\sqrt{\alpha_L} = \sqrt{\frac{2-w}{4N(1-w)} + 1} - \sqrt{\frac{w}{4N(1-w)}}. \quad (\text{A12})$$

Hence the lower bound for  $R_{\rho,w}$  is obtained as

$$\min_{\alpha} B_w = \begin{cases} \sqrt{w(1-w)\left(N + \frac{1}{4}\right) + \frac{w}{4} - \frac{1}{2}} & \text{if } N > N_L \\ 0 & \text{if } N \leq N_L, \end{cases} \quad (\text{A13})$$

where  $N_L = (1-w)/4w$ .

As stated in Sec. II B, in order to use these results in the two-mode separability problem the bound should fulfill two criteria (i)  $L(N) + wN^2$  should be concave, (ii)  $L(N_1) + L(N_2) \geq L(N_1 + N_2) + L(0)$ . Equation (A13) does not fulfill (ii), thus a weaker bound satisfying both conditions has to be chosen.  $L_w(N)$  as defined in Eq. (17) is such a bound. It coincides with Eq. (A13) for  $N \geq N_L$  while for  $N < N_L$  it is a linear function of  $N$  and it is negative. The function giving  $L_w(N)$  for  $N > N_L$  [top line in Eq. (A13)] cannot simply be extended to  $N \leq N_L$  as it was done in part 1 of the Appendix for the simpler uncertainty relation, since in this case (i) would not be satisfied.

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 [18] The following deduction can be generalized for the sum of uncertainties of a Hermitian ( $X$ ) and a non-Hermitian ( $Y=Q+iP$ ) operator as  $(\Delta_\rho X)^2 + (\Delta_\rho Y)^2 \geq |\langle [X, Y] \rangle_\rho| + i\langle [P, Q] \rangle_\rho$ . (Here  $P$  and  $Q$  are Hermitian operators.) If both  $X$  and  $Y$  are Hermitian (i.e.,  $P=0$ ) the inequality reduces to  $(\Delta_\rho X)^2 + (\Delta_\rho Y)^2 \geq |\langle [X, Y] \rangle_\rho|$ . This latter expression can also be obtained from the usual product form of the Heisenberg uncertainty  $(\Delta_\rho X)^2 (\Delta_\rho Y)^2 \geq |\langle [X, Y] \rangle_\rho|^2 / 4$  using the inequality  $(\Delta_\rho X)^2 + (\Delta_\rho Y)^2 \geq 2\sqrt{(\Delta_\rho X)^2 (\Delta_\rho Y)^2}$ .