Multipartite entanglement and high precision metrology

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Outline

1. **Motivation**
   - Why many-body entanglement is important?
2. **Different types of multipartite entanglement**
   - Multipartite entanglement
3. **Entanglement detection in systems with very many particles**
   - Physical systems
   - Spin squeezing
   - Generalized spin squeezing
4. **Fisher information and entanglement**
   - Quantum Fisher information
   - Properties of the Quantum Fisher information
   - Quantum Fisher information and entanglement
   - Points in the 3D space

Why is multipartite entanglement interesting?

- There have been many experiments recently aiming to create many-body entangled states.
- Quantum Information Science can help to find good targets for such experiments.
- Multipartite entangled states are needed in applications such as metrology.

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Several qubits

Definition

A state is (fully) separable if it can be written as $\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}$.

Definition

A pure multi-qubit quantum state is called biseparable if it can be written as the tensor product of two multi-qubit states

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle.$$

Here $|\Psi\rangle$ is an N-qubit state. A mixed state is called biseparable, if it can be obtained by mixing pure biseparable states.

Definition

If a state is not biseparable then it is called genuine multi-partite entangled.

k-producibility/k-entanglement

Definition

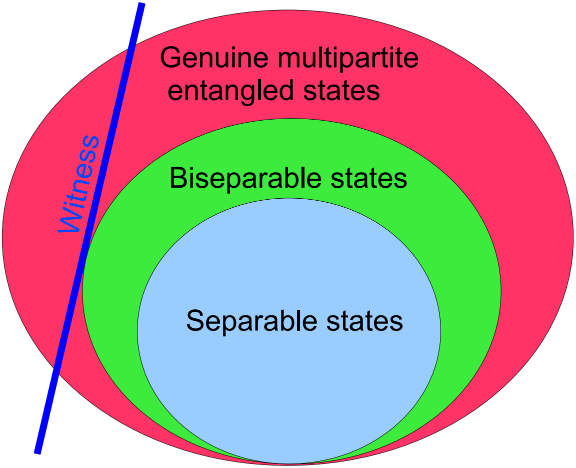
A pure state is *k-producible or k-entangled* if it can be written as

$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

where $|\Phi_l\rangle$ are states of at most k qubits. A mixed state is k-producible, if it is a mixture of k-producible pure states.

Convex sets for the multipartite case

- The idea of convex sets also work for the multi-qubit case: A state is biseparable if it can be obtained by mixing pure biseparable states.



Examples

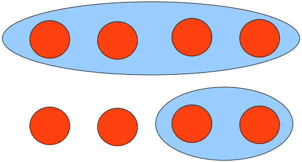
Examples

Two entangled states of four qubits:

$$|GHZ_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|\Psi_B\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |0011\rangle) = \frac{1}{\sqrt{2}}|00\rangle \otimes (|00\rangle + |11\rangle).$$

- The first state is genuine multipartite entangled, the second state is biseparable.



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Physical systems

State-of-the-art in experiments

- 100,000 atoms realizing an array of 1D Ising spin chains (Nature, 2003)
- Spin squeezing with 10^6 - 10^{12} atoms (Nature, 2001)

Main challenge

- The particles cannot be addressed individually.
- Only collective quantities can be measured.
- New type of entangled states and entanglement criteria are needed.

Many-particle systems

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We can also measure the

$$(\Delta J_l)^2 := \langle J_l^2\rangle - \langle J_l\rangle^2$$

variances.

Outline

Spin squeezing

Definition

Spin squeezing criterion for the detection of quantum entanglement

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}.$$

If a quantum state violates this criterion then it is entangled.

- Application: Quantum metrology, magnetometry.

[A. Sørensen *et al.*, Nature **409**, 63 (2001)]

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Complete set of the generalized spin squeezing criteria

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$

$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

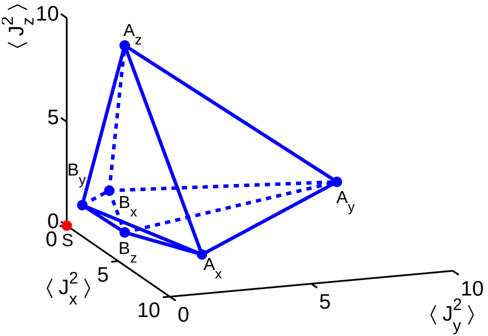
- Then any state violating the following inequalities is entangled

$$\begin{aligned}
\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle &\leq N(N+2)/4, \\
(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq N/2, \\
\langle J_k^2 \rangle + \langle J_l^2 \rangle - N/2 &\leq (N-1)(\Delta J_m)^2, \\
(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] &\geq \langle J_m^2 \rangle + N(N-2)/4.
\end{aligned}$$

where k, l, m takes all the possible permutations of x, y, z.
[GT, C. Knapp, O. Gühne, and H.J. Briegel, Phys. Rev. Lett. 2007]

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J_{x/y/z}^2 \rangle$ space.
- Separable states correspond to points inside the polytope. Note: Convexity comes up again!



The derivation of such criteria

- The derivation of such criteria is partly based on entanglement detection with uncertainty relations.
- For a multi-qubit pure product state $|\Psi_{\mathrm{P}}\rangle = \bigotimes_k |\psi_k\rangle$we have

$$(\Delta J_l)^2 = \sum_k (\Delta j_l^{(k)})^2{}_{\psi_k}.$$

- Hence,

$$\sum_{l=x,y,z} (\Delta J_l)^2_{|\Psi_{\mathrm{P}}\rangle} = \sum_{l=x,y,z} \sum_{k=1}^{N} (\Delta J_l)^2_{|\Psi_k\rangle} =$$

$$\frac{1}{4}\sum_{k=1}^{N}(3 - \langle\sigma_x^{(k)}\rangle^2 - \langle\sigma_y^{(k)}\rangle^2 - \langle\sigma_z^{(k)}\rangle^2) = \frac{N}{2}.$$

- Due to the concavity of the variance, for mixed separable states we have

$$\sum_{l=x,y,z} (\Delta J_l)^2 \geq \frac{N}{2}.$$

Outline

Quantum Fisher information

- One of the important applications of entangled multipartite quantum states is sub-shotnoise metrology.
  [V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).]
- Multipartite entanglement, not simple nonseparability, is needed for extreme spin squeezing, which can be applied in spectroscopy and atomic clocks.
  [A.S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001).]
- Not all entangled states are useful for phase estimation, at least in a linear interferometer.
  [P. Hyllus, O. Gühne, and A. Smerzi, arXiv:0912.4349.]

Quantum Fisher information II

Quantum Cramér-Rao bound

For such a linear interferometer the phase estimation sensitivity is limited by the Quantum Cramér-Rao bound as

$$\Delta\theta \geq \frac{1}{\sqrt{F_Q[\varrho, J_{\vec{n}}]}},$$

where F_Q is the quantum Fisher information, ϱ is a quantum state and $J_{\vec{n}}$ is a component of the collective angular momentum in the direction $\vec{n}$. The phase estimation is connected to the dynamics $\varrho = e^{-i\theta J_{\vec{n}}}\varrho_0 e^{+i\theta J_{\vec{n}}}$.

[C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976);
A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982).]

Quantum Fisher information III

- The quantum Fisher information is the supremum of the following [Braunstein, Caves, 1994]

$$F(\varrho(\theta),\{E(\xi)\}) = \int \frac{[\mathrm{Tr}\varrho(\theta)' E(\xi)]^2}{\mathrm{Tr}\varrho(\theta)E(\xi)} d\xi.$$

- In another context, there are several possible Fisher informations. The Braunstein-Caves's one the *minimal* Fisher information.
- Notation:

$$\varrho(\theta) = D + B\theta,$$

such that $\mathrm{Tr}(B) = 0$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, ...)$.

$$F_{\min}(D,B) = \sum_{ij} \frac{2}{\lambda_i + \lambda_j} |B_{ij}|^2 \qquad F_{\min}(D, i[D,X]) = \sum_{ij} \frac{2(\lambda_i - \lambda)^2}{\lambda_i + \lambda_j} |X_{ij}|^2.$$

[D. Petz, Monotone metrics on matrix spaces, Linear Algebra Appl. 244(1996), 81–96;
D. Petz and Cs. Sudár, World Scientific, 1999;
D. Petz and C. Ghinea, arXiv:1008.2417.]

Quantum Fisher information IV

Notation

The two notations are equivalent

$$F[\varrho, X] \equiv F_{\min}(\varrho, i[\varrho, X]).$$

Outline

Properties of the Quantum Fisher information

For calculating many quantities, it is sufficient to know that following two relations.

1. For a pure state ϱ, we have $F[\varrho, J_l] = 4(\Delta J_l)^2_\varrho$.
2. $F[\varrho, J_l]$ is convex in the state, that is
   $F[p_1\varrho_1 + p_2\varrho_2, J_l] \le p_1 F[\varrho_1, J_l] + p_2 F[\varrho_2, J_l]$.

From these two statements, it also follows that $F[\varrho, J_l] \le 4(\Delta J_l)^2_\varrho$.

[C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976);

A. S. Holevo, *Probabilistic and Statistical Aspect of Quantum Theory* (North-Holland, Amsterdam, 1982);

S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 72, 3439 (1994); L. Pezzé and A. Smerzi, Phys. Rev. Lett. 102, 100401 (2009).]

Properties of the Quantum Fisher information II

For computing the Fisher information numerically, we recall that the quantum Fisher information $F_Q[\varrho, J_{\vec{n}}]$ for any $\vec{n}$ can be given as

$$F_Q[\varrho, J_{\vec{n}}] = 4\vec{n}^T \Gamma_C \vec{n}.$$

Here, the Γ_C matrix is defined as

$$[\Gamma_C]_{ij} = \frac{1}{2}\sum_{l,m}(\lambda_l + \lambda_m)\left(\frac{\lambda_l - \lambda_m}{\lambda_l + \lambda_m}\right)^2 \langle l|J_i|m\rangle\langle m|J_j|l\rangle,$$

where the sum is over the terms for which $\lambda_l + \lambda_m \neq 0$, and the density matrix has the decomposition

$$\varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

For pure states, and $[\Gamma_C]_{ij} = \langle J_i J_j + J_j J_i\rangle/2 - \langle J_i J_j\rangle$.

[P. Hyllus, O. Gühne, and A. Smerzi, arXiv:0912.4349.]

Outline

Quantum Fisher information and entanglementand

Pezzé, Smerzi, PRL 2009

For N-qubit separable states, the values of $F_Q[\varrho, J_l]$ for $l = x, y, z$ are bounded as

$$F_Q[\varrho, J_l] \leq N.$$

Here, $J_l = \frac{1}{2}\sum_{k=1}^{N}\sigma_l^{(k)}$ where $\sigma_l^{(k)}$ are the Pauli spin matrices for qubit (k).

Quantum Fisher information and entanglement II

Observation 1

For N-qubit separable states, the values of $F_Q[\varrho, J_l]$ for $l = x, y, z$ are bounded as

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq 2N. \qquad (2)$$

- Later we will also show that Eq. (2) is a condition for the average sensitivity of the interferometer. All states violating Eq. (2) are entangled.

Quantum Fisher information and entanglement III

Observation 2

For quantum states, the Fisher information is bounded by above as

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq N(N+2). \tag{3}$$

Greenberger-Horne-Zeilinger (GHZ) states and N-qubit symmetric Dicke states with $\frac{N}{2}$ excitations saturate Eq. (3).

- The above symmetric Dicke state has been investigated recently due to its interesting entanglement properties. It has also been noted that above Dicke state gives an almost maximal phase measurement sensitivity in two orthogonal directions.
- In general, pure symmetric states for which $\langle J_l \rangle = 0$ for $l = x, y, z$ saturate Eq. (3).

Quantum Fisher information and multipartite entanglement

Next, we will consider k-producible or k-entangled states:

***Observation* 3**

For N-qubit k-producible states states, the sum of three Fisher information terms is bounded from above by

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq nk(k+2) + (N-nk)(N-nk+2).$$

where n is the integer part of $\frac{N}{k}$. For the $k = N-1$ case, this bound can be improved

$$\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq N^2 + 1. \tag{4}$$

Eq. (4) is also the inequality for biseparable states. Any state that violates Eq. (4) is genuine multipartite entangled.

Quantum Fisher information and multipartite entanglement

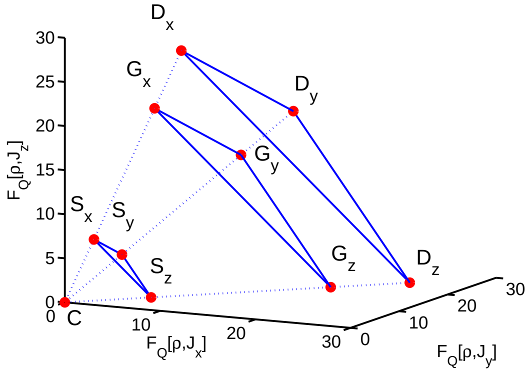


Figure: Interesting points in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space for $N = 6$ particles. Points corresponding to separable states satisfy Eq. (2) and are not above the $S_x - S_y - S_z$ plane. Biseparable states satisfy Eq. (4) and are not above the $G_x - G_y - G_z$ plane.

Proof of Observation 1

First, we shown that Observation 1 is true for pure states. For every N-qubit pure product state of the form

$$|\Psi_{\rm P}\rangle = \otimes_{k=1}^{N} |\Psi_k\rangle$$

we have

$$\sum_{l=x,y,z} (\Delta J_l)^2_{|\Psi_{\rm P}\rangle} = \sum_{l=x,y,z} \sum_{k=1}^{N} (\Delta J_l)^2_{|\Psi_k\rangle} =$$

$$\frac{1}{4} \sum_{k=1}^{N} (3 - \langle \sigma_x^{(k)} \rangle^2 - \langle \sigma_y^{(k)} \rangle^2 - \langle \sigma_z^{(k)} \rangle^2) = \frac{N}{2}.$$

For mixed states, $\sum_{l=x,y,z} F_Q[\varrho, J_l] \leq 2N$ follows from the convexity of the Fisher information. This finishes the proof.
[G. Tóth, Phys. Rev. A 69, 052327 (2004).]

Proof of Observation 1 - II

- Observation 1 can be reformulated with the eigenvalues of Γ_C as

$$\mathrm{Tr}(\Gamma_C) \quad \leq \quad 2N.$$

- We can rewrite the left hand side as

$$\mathrm{avg}_{\vec{n}}(F_Q[\varrho, J_{\vec{n}}]) = 4\mathrm{avg}_{\vec{n}}\{\mathrm{Tr}[\Gamma_C(\vec{n}\vec{n}^T)]\} = 4\mathrm{Tr}(\Gamma_C \frac{\mathbb{1}}{3}),$$

where averaging is over all three-dimensional unit vectors. Thus, Observation 1 is a condition for the average sensitivity of the interferometer.

Proof of Observation 2

- Again, we have to use

$$\sum_{l=x,y,z} F(\varrho, J_l) \le 4 \sum_{l=x,y,z} (\Delta J_l)^2 \le N(N+2). \qquad (5)$$

- For pure states, the first inequality of Eq. (5) is saturated.
- For symmetric states with $\langle J_l \rangle = 0$ for $l = x, y, z$, the second inequality is saturated. Hence GHZ states and Dicke states with $\frac{N}{2}$ excitations saturate Eq. (5).

Outline

Which part of the space corresponds to quantum states

- We discuss which part of the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space contains points corresponding to states with different degree of entanglement.
- This is important, since apart from finding inequalities for states of various types of entanglement, we have to show that there are states that fulfill these inequalities.

Which part of the space corresponds to quantum states

Let us see first the interesting points of the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space and the corresponding quantum states:

- A completely mixed state

$$\varrho_C = \frac{\mathbb{1}}{2^N}.$$

  corresponds to the point $C(0,0,0)$.

- States corresponding to the points $S_x(0,N,N), S_y(N,0,N), S_z(0,N,N)$ are

$$|\Psi\rangle_{S_l} = |+\frac{1}{2}\rangle_l^{\otimes N/2} \otimes |-\frac{1}{2}\rangle_l^{\otimes N/2}$$

  for $l = x, y, z$.

Which part of the space corresponds to quantum states II

- For the point $D_z(N(N+2)/2, N(N+2)/2, 0)$, a corresponding quantum state is an N-qubit symmetric Dicke state with $N/2$ excitations in the z basis.

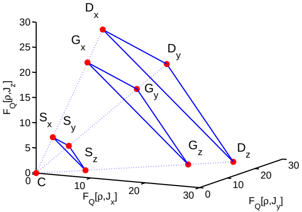
$$|\mathcal{D}_N^{(N/2)}\rangle = \binom{N}{N/2}^{-\frac{1}{2}} \sum_k \mathcal{P}_k\{|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\},$$

  where $\sum_k \mathcal{P}_k$ denotes summation over all possible permutations.

- For the point (N, N, N^2), a corresponding quantum state is an N-qubit GHZ states in the z basis

$$|\Psi\rangle_{GHZ_z} = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right).$$

Which part of the space corresponds to quantum states III



- For all points in the S_x, S_y, S_z polytope, there is a corresponding pure product state for even N.
- For given $F[\varrho, J_l]$ for $l = x, y, z$, such a state is defined as

$$\varrho = \left[\frac{\mathbb{1}}{4} + \frac{1}{4}\sum_{l=x,y,z} c_l \sigma_l\right]^{\otimes N/2} \otimes \left[\frac{\mathbb{1}}{4} - \frac{1}{4}\sum_{l=x,y,z} c_l \sigma_l\right]^{\otimes N/2},$$

where $c_l^2 = 1 - \frac{F_Q[\varrho, J_l]}{N}$, where $\sum_l c_l^2 = 1$.

Which part of the space corresponds to quantum states IV

- For all points in the D_x, D_y, D_z polytope, there is a corresponding quantum state if N is divisible by 4. To see this, let us consider the following quantum states for even N

$$|\Psi_{\text{even}}\rangle = \sum_{n=0,2,4,\ldots,N/2} c_n \frac{1}{\sqrt{2}} \left(|\mathcal{D}_N^{(n)}\rangle + |\mathcal{D}_N^{(N-n)}\rangle \right), \tag{6}$$

  where c_n are complex coefficients. For $|\Psi_{\text{even}}\rangle$, we have $\langle J_l \rangle = 0$ for $l = x, y, z$. Finally, $\langle J_l J_m + J_m J_l \rangle = 0$ if $l \neq m$, thus for $|\Psi_{\text{even}}\rangle$ the matrix Γ_C is diagonal.

- For the case of N is a multiple of 4, one can consider the states of the form

$$|\Psi(\alpha, \beta, \gamma)\rangle = \alpha_x |\mathcal{D}_N^{(N/2)}\rangle_x + \alpha_y |\mathcal{D}_N^{(N/2)}\rangle_y + \alpha_z |\mathcal{D}_N^{(N/2)}\rangle_z, \tag{7}$$

  where α_l are complex coefficients. (Note that $|\mathcal{D}_N^{(N/2)}\rangle_l$ are not pairwise orthogonal.) The states (7) fill the polytope D_x, D_x, and D_z.

Which part of the space corresponds to quantum states V

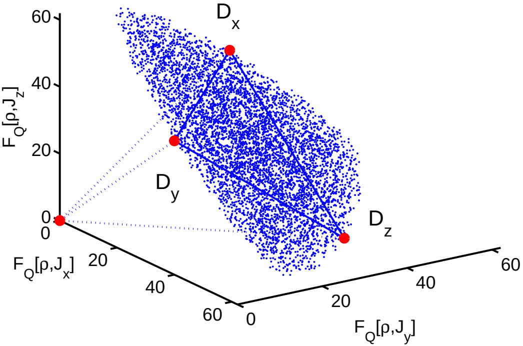


Figure: Randomly chosen points in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])$-space corresponding to states of the form $|\Psi(\alpha,\beta,\gamma)\rangle = \alpha_x|\mathcal{D}_N^{(N/2)}\rangle_x + \alpha_y|\mathcal{D}_N^{(N/2)}\rangle_y + \alpha_z|\mathcal{D}_N^{(N/2)}\rangle_z$, for $N = 8$. All the points are in the plane of D_x, D_y and D_z.

Which part of the space corresponds to quantum states VI

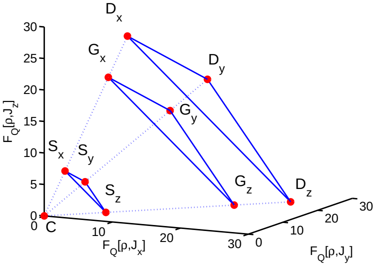
- Three-dimensional polytopes. The points corresponding to the mixed state are on a curve in the $(F_Q[\varrho, J_x], F_Q[\varrho, J_y], F_Q[\varrho, J_z])\text{-space}$. In the general case, this curve is not a straight line. For the case of mixing a pure state with the completely mixed state the curve is a straight line. Such a state is defined as

$$\varrho^{(\text{mixed})}(p) = p\varrho + (1-p)\frac{\mathbb{1}}{2^N}$$

- Using the formula for Γ_C, after simple calculations we have

$$\Gamma_C^{(\text{mixed})}(p) = \frac{p^2}{p + (1-p)2^{-(N-1)}}\Gamma_C^{(\varrho)}.$$

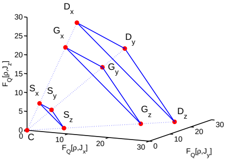
Which part of the space corresponds to quantum states VII



Observation 5. If N is even, then there is a separable state for each point in the S_x, S_y, S_z, C polytope.

Proof. This is because there is a pure product state corresponding to any point in the S_x, S_y, S_z polytope. When mixing any of these states with the completely mixed state, we obtain states that correspond to points on the line connecting the pure state to point C.

Which part of the space corresponds to quantum states VIII



Observation 6. If N is divisible by 4, then for all the points of the $D_x, D_y, D_z, G_x, G_y, G_z$ polytope, there is a quantum state with genuine multipartite entanglement.

Proof. There is a quantum state for all points in the D_x, D_y, D_z polytope. Mixing them with the completely mixed state, states corresponding to all points of the C, D_x, D_y, D_z polytope can be obtained. Based on Observation 2, states corresponding to the points in the $D_x, D_y, D_z, G_x, G_y, G_z$ polytope are genuine multipartite entangled.

Finally, note that all the quantum states we presented in this section have a diagonal Γ_C matrix.

Summary

- We discussed entanglement detection in multipartite systems.
- We considered
  - entanglement detection with variances and the Fisher information
  - we considered different types of multipartite entanglement

GT, “Multipartite entanglement and high precision metrology”,
arxiv:1006.4368.

See another article on a similar topic:
P. Hyllus et al., arXiv:1006.4366.

THANK YOU FOR YOUR ATTENTION!