Optimal spin squeezing inequalities detect bound entanglement in spin models quant-ph/0702219

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Outline

1. Motivation
2. Entanglement detection with collective observables
3. Complete characterization of separable states
4. Multipartite bound entanglement in spin models

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Motivation

- In many quantum control experiments the qubits cannot be individually accessed. We still would like to detect entanglement.
- The spin squeezing criterion is already known. It would be interesting to find similar criteria that detect entanglement in the vicinity of useful quantum states.
- It would be interesting to obtain a complete characterization of separable states at least concerning the expectation values and variances of collective observables. This would help to move towards the solution of the separability problem.
- It would be interesting to find criteria detecting bound entanglement. In our case: Entanglement that is PPT with respect to all bipartitions.

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Spin squeezing

1. Spin squeezing, according to the original definition, is interpreted in the following context. The variances of the angular momentum components are bounded by the following uncertainty relation

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle Jz\rangle|^2. \tag{1}$$

If $(\Delta J_x)^2$ is smaller than the standard quantum limit $|\langle Jz\rangle|/2$ then the state is called spin squeezed.

2. In practice this means that the angular momentum of the state has a small variance in one direction, while in an orthogonal direction the angular momentum is large.

[M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993).]

Definition of entanglement

- Fully separable states are states that can be written in the form
$$\rho = \sum_l p_l \rho_l^{(1)} \otimes \rho_l^{(2)} \otimes ... \otimes \rho_l^{(N)}, \quad (2)$$
where $\sum_l p_l = 1$ and $p_l > 0$.

- A state is entangled if it is not separable.

- Note that one could also look for other type of entanglement in many-particle systems, e.g., entanglement in the two-qubit reduced density matrix.

Spin squeezing and entanglement

- What can we measure if we cannot access the qubits individually? We can measure the expectation values of the collective angular momentum components

$$J_{x/y/z} := \frac{1}{2} \sum_{k=1}^{N} \sigma_{x/y/z}^{(k)}, \tag{3}$$

where $\sigma_{x/y/z}^{(k)}$ are Pauli matrices. We can also measure the variances $(\Delta J_{x/y/z})^2$. [Here $(\Delta A)^2 := \langle A^2 \rangle - \langle A \rangle^2$.]

- The spin squeezing criteria for entanglement detection is

$$\frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq \frac{1}{N}. \tag{4}$$

If it is violated then the state is entangled.

[A. Sørensen *et al.*, Nature **409**, 63 (2001).]

Generalized spin squeezing entanglement criteria

- Criterion 1. For separable states $\langle J_x^2\rangle + \langle J_y^2\rangle \leq (N^2+N)/4$ holds. This can be used to detect entanglement close to N-qubit symmetric Dicke states with $N/2$ excitations. [G. Tóth, J. Opt. Soc. Am. B **24**, 275 (2007).]

- Criterion 2. Separable states must fulfill $(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq N/2$. It is maximally violated by a many-body singlet, e.g., the ground state of an anti-ferromagnetic Heisenberg chain. [G. Tóth, Phys. Rev. A **69**, 052327 (2004).]

- Criterion 3. For symmetric separable states $1 - 4\langle J_m\rangle^2/N^2 \leq 4(\Delta J_m)^2/N$ holds. [J. Korbicz *et al.* Phys. Rev. Lett. **95**, 120502 (2005).]

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Complete characterization of separable states

- Let us assume that for a physical system the values of $\mathbf{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$ and $\langle J^2_{x/y/z} \rangle$ are known. If the system is in a separable state, the following inequalities hold:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq N(N+2)/4, \tag{5a}$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq N/2, \tag{5b}$$

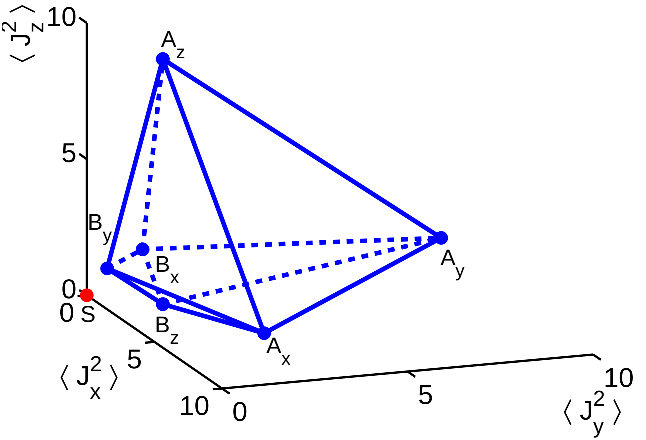
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle - N/2 \leq (N-1)(\Delta J_m)^2, \tag{5c}$$

$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + N(N-2)/4, \tag{5d}$$

where k, l, m take all the possible permutations of x, y, z.

The polytope

- The previous inequalities, *for fixed* $\langle J_{x/y/z} \rangle$, describe a polytope in the $\langle J^2_{x/y/z} \rangle$ space. The polytope has six extreme points: $A_{x/y/z}$ and $B_{x/y/z}$. For $\langle \mathbf{J} \rangle = 0$ and $N = 6$ the polytope is the following:



The polytope II

- The coordinates of the extreme points are

$$A_x := \left[\frac{N^2}{4} - \kappa(\langle J_y\rangle^2 + \langle J_z\rangle^2), \frac{N}{4} + \kappa\langle J_y\rangle^2, \frac{N}{4} + \kappa\langle J_z\rangle^2\right],$$

$$B_x := \left[\langle J_x\rangle^2 + \frac{\langle J_y\rangle^2 + \langle J_z\rangle^2}{N}, \frac{N}{4} + \kappa\langle J_y\rangle^2, \frac{N}{4} + \kappa\langle J_z\rangle^2\right],$$

  where $\kappa := (N-1)/N$. The points $A_{y/z}$ and $B_{y/z}$ can be obtained from these by permuting the coordinates.

- Now it is easy to prove that an inequality is a necessary condition for separability: All the six points must satisfy it.

The polytope III

- Let us take the $\langle \mathbf{J} \rangle = 0$ case first.
- Then the state corresponding to A_x is the equal mixture of

$$|+1, +1, +1, +1, ...\rangle_x \tag{6}$$

and

$$|-1, -1, -1, -1, ...\rangle_x. \tag{7}$$

- The state corresponding to B_x is

$$|+1\rangle_x^{\otimes N/2} \otimes |-1\rangle_x^{\otimes N/2}. \tag{8}$$

- Separable states corresponding to $A_{y/z}$ and $B_{y/z}$ are defined similarly.

The polytope IV

- General case: $\langle \mathbf{J} \rangle \neq 0$.
- A separable state corresponding to A_x is

$$\rho_{A_x} = p(|\psi_+\rangle\langle\psi_+|)^{\otimes N} + (1-p)(|\psi_-\rangle\langle\psi_-|)^{\otimes N}. \tag{9}$$

  Here $|\psi_{+/-}\rangle$ are the single qubit states with Bloch vector coordinates $(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) = (\pm c_x, 2\langle J_y\rangle/N, 2\langle J_z\rangle/N)$ where $c_x := \sqrt{1 - 4(\langle J_y\rangle^2 + \langle J_z\rangle^2)/N^2}$. The mixing ratio is defined as $p := 1/2 + \langle J_x\rangle/(Nc_x)$.
- If $N_1 := Np$ is an integer, we can also define the state corresponding to the point B_x as

$$|\phi_{B_x}\rangle = |\psi_+\rangle^{\otimes N_1} \otimes |\psi_-\rangle^{\otimes (N-N_1)}. \tag{10}$$

  If N_1 is not an integer then one can find a point B'_x such that such that its distance from B_x is smaller than $1/4$.

In what sense is the characterization complete?

- For any value of $\mathbf{J}$ there are separable states corresponding to $A_{x/y/z}$.
- For certain values of $\mathbf{J}$ and N (e.g., $\mathbf{J} = 0$ and even N) there are separable states corresponding to points $B_{x/y/z}$.
- However, there are always separable states corresponding to points $B'_{x/y/z}$ such that their distance from $B_{x/y/z}$ is smaller than $1/4$.
- In the limit $N \to \infty$ for a fixed normalized angular momentum $2\mathbf{J}/N$ the difference between the volume of our polytope and the volume of set of points corresponding to separable states decreases with N as $\Delta V/V \propto N^{-2}$, hence in the macroscopic limit the characterization is complete.

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Two-qubit entanglement

- Our criteria can detect entangled states for which the reduced two-qubit density matrix is separable.

- This might look surprising since all our criteria contain operator expectation values that can be computed knowing the average two-qubit density matrix

$$\rho_{12} := \frac{1}{N(N-1)} \sum_{k \neq l} \rho_{kl}, \tag{11}$$

  and no information on higher order correlation is used.

- Still, our criteria do not merely detect entanglement in the reduced two-qubit state!

Two-qubit entanglement II

- Two-qubit symmetric separable states have the form

$$\rho_{12} = \sum_k p_k \rho_k \otimes \rho_k. \qquad (12)$$

  For such states it is always possible to find an N-qubit separable state, which has ρ_{12} as it reduced state:

$$\rho = \sum_k p_k \rho_k \otimes \rho_k \otimes ... \otimes \rho_k. \qquad (13)$$

- However, there are two-qubit separable states for which this is not possible. For example, these can be of the form

$$\rho_{12} = \frac{1}{2}(\rho_1 \otimes \rho_2 + \rho_2 \otimes \rho_1). \qquad (14)$$

  Clearly, it is not easy to find an N-qubit state for such a state.

Bound entanglement in spin chains

- Let us consider four spin-1/2 particles, interacting via the Hamiltonian

$$H = (h_{12} + h_{23} + h_{34} + h_{41}) + J_2(h_{13} + h_{24}), \qquad (15)$$

  where $h_{ij} = \sigma_x^{(i)} \otimes \sigma_x^{(j)} + \sigma_y^{(i)} \otimes \sigma_y^{(j)} + \sigma_z^{(i)} \otimes \sigma_z^{(j)}$ is a Heisenberg interaction between the qubits i, j.

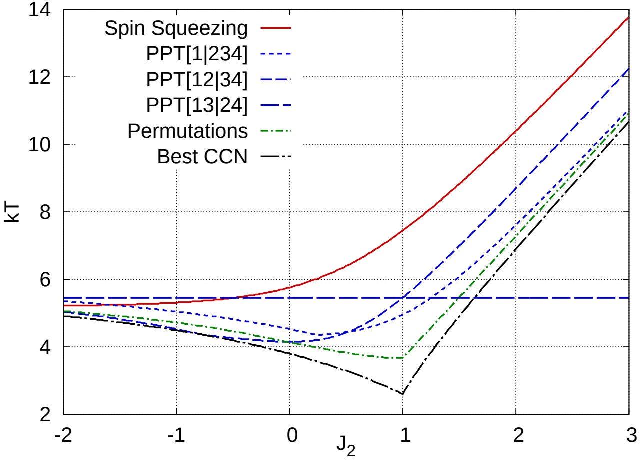
- For the above Hamiltonian we compute the thermal state $\varrho(T, J_2) \propto \exp(-H/kT)$ and investigate its separability properties.

- For different separability criteria we calculate the maximal temperature, below which the criteria detect the states as entangled.

- For $J_2 \gtrsim -0.5$, the spin squeezing inequality is the strongest criterion for separability. It allows to detect entanglement even if the state has a positive partial transpose (PPT) with respect to all bipartition.

Bound entanglement in spin chains II

- Bound temperatures for entanglement



Bound entanglement in spin chains III

- We found bound entanglement that is PPT with respect to all bipartitions in XY and Heisenberg chains, and also in XY and Heisenberg models on a completely connected graph, up to 10 qubits.
- Thus for these models, which appear in nature, there is a considerable temperature range in which the system is already PPT but not yet separable.
- One can expect the same also for the thermodynamic limit.

Conclusions

- We presented a family of entanglement criteria that are able to detect any entangled state that can be detected based on the first and second moments of collective angular momenta.
- We explicitly determined the polytope corresponding to separable states in the space of second order moments.
- We applied our findings to examples of spin models, showing the presence of bound entanglement in these models.

*** THANK YOU ***