Entanglement detection using the stabilizer theory

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G. Tóth and O. Gühne, PRL **94**, 060501 (2005).

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Motivation

In many-qubit systems state tomography is not feasible since the number of measurements needed increases exponentially with the size of the system.

We can still expect to do the following things

- Decide whether the state is entangled
- Measure the fidelity with respect to a given state
- Decide whether the state is genuine multi-qubit entangled

Problem: with usual methods even for these tasks the number of measurements increases exponentially with the system size, if only *local* masurements are allowed.

Outline

- Stabilizer theory (GHZ and cluster states)
- Detecting entanglement with few measurements
- Finding a lower bound on the fidelity with few measurements
- Detecting genuine multi-qubit entanglement with few measurements

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Stabilizer theory

Definition: A quantum state $|\Psi\rangle$ is stabilized by an operator *S* if

$$S|\Psi\rangle = |\Psi\rangle.$$

In other words, *S* is the stabilizing operator of $|\Psi\rangle$.

Stabilizer theory is used in quantum error correction and fault tolerant quantum computation.

The key idea is that an *N*-qubit quantum state can uniquely be defined by *N* stabilizing operators. For certain quantum states these operators are very simple ...

D. Gottesmann, PRA. **54**, 1862 (1996).

GHZ states

Generalized *N*-qubit GHZ state:

$$|GHZ_N\rangle = \frac{1}{\sqrt{2}}\left(|000...00\rangle + |111...11\rangle\right)$$

Stabilizing operators of a GHZ state:

$$S_1^{(GHZ_N)} = X^{(1)}X^{(2)}\cdots X^{(N)},$$

$$S_2^{(GHZ_N)} = Z^{(1)}Z^{(2)},$$

$$S_3^{(GHZ_N)} = Z^{(2)}Z^{(3)}, \quad \ldots$$

$$S_N^{(GHZ_N)} = Z^{(N-1)}Z^{(N)}.$$

Not only these operators, but also their products stabilize the GHZ state. These form a group called *stabilizer*. S_k's are the generators of the stabilizer.

GHZ states – The stabilizer group

Three-qubit example: the stabilizer group has 8 elements

Generators:

$$X^{(1)} X^{(2)} X^{(3)}$$

$$Z^{(1)} Z^{(2)}$$

$$Z^{(2)} Z^{(3)}$$

Obtained from products of the generators:

$$-Y^{(1)} X^{(2)} Y^{(3)}$$

$$-Y^{(1)} Y^{(2)} X^{(3)}$$

$$-X^{(1)} Y^{(2)} Y^{(3)}$$

$$Z^{(1)} Z^{(3)}$$

$$1$$

Cluster states

Cluster states appear in

- error correction, fault tolerant quantum computing, and
- measurement based quantum computing.
- Naturally arise in spin chains.

Stabilizing operators of an *N*-qubit cluster state $|C_N\rangle$

$$S_k^{(C_N)} = Z^{(k-1)} X^{(k)} Z^{(k+1)},$$

where *k*=1,2,...,*N* and

$$Z^{(0)} = Z^{(N+1)} = 1.$$

R. Raussendorf and H.J. Briegel, PRL **86**, 5188 (2001).
See also recent experiments with a four-qubit cluster state with photons in Vienna and at MPQ, Garching.

- Stabilizer theory (GHZ and cluster states)
- **Detecting entanglement with few measurements**
- Finding a lower bound on the fidelity with few measurements
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Entanglement criterion I

Let us construct an entanglement criterion with the stabilizing operators of the GHZ state.

Criterion 1: For fully separable states

$$\left\langle S_1^{(GHZ_N)} \right\rangle + \left\langle S_l^{(GHZ_N)} \right\rangle \le 1,$$

where $2 \le l \le N.$

Proof. For product states, using the Cauchy-Schwarz ineq.

$$\left\langle S_1^{(GHZ_N)} \right\rangle + \left\langle S_l^{(GHZ_N)} \right\rangle = \left\langle X^{(1)} X^{(2)} \cdots X^{(N)} \right\rangle + \left\langle Z^{(l-1)} Z^{(l)} \right\rangle =$$

$$\left\langle X^{(1)} \right\rangle \left\langle X^{(2)} \right\rangle \cdots \left\langle X^{(N)} \right\rangle + \left\langle Z^{(l-1)} \right\rangle \left\langle Z^{(l)} \right\rangle \le$$

$$\left| \left\langle X^{(l-1)} \right\rangle \right| \left| \left\langle X^{(l)} \right\rangle \right| + \left| \left\langle Z^{(l-1)} \right\rangle \right| \left| \left\langle Z^{(l)} \right\rangle \right| \le 1.$$

It is easy to see that this is also true for mixed separable states.

Entanglement criterion II

Let us look at the previous condition:

$$\left\langle S_1^{(GHZ_N)} \right\rangle + \left\langle S_l^{(GHZ_N)} \right\rangle \leq 1, \qquad l = 2, 3, ..., N.$$

The left hand side is maximal for the GHZ state, i.e.,
this criterion detects states close the GHZ state.

Robustness to noise: Let us consider a noisy GHZ state of the form.

$$\rho(p_{\text{noise}}) = p_{\text{noise}} \frac{1}{2^N} + (1 - p_{\text{noise}}) \left| GHZ_N \right\rangle \left\langle GHZ_N \right|.$$

How much noise is allowed by our method before it stops detecting the state as entangled? Answer:

$$p_{\text{noise}} < \frac{1}{2}.$$

Entanglement criterion III

Generalization. Choosing any two locally non-commuting stabilizing operators we can construct a necessary condition for separability

$$\left\langle S_k^{(GHZ_N)} \right\rangle + \left\langle S_l^{(GHZ_N)} \right\rangle \le 1.$$

If it is violated then the system is entangled.

Three-qubit examples:

$$\left\langle X^{(1)} X^{(2)} X^{(3)} \right\rangle + \left\langle Z^{(1)} Z^{(2)} \right\rangle \le 1,$$

$$-\left\langle Y^{(1)} Y^{(2)} X^{(3)} \right\rangle + \left\langle Z^{(1)} Z^{(2)} \right\rangle \le 1,$$

$$\left\langle X^{(1)} X^{(2)} X^{(3)} \right\rangle - \left\langle Y^{(1)} Y^{(2)} X^{(3)} \right\rangle \le 1,$$

Comparison with Bell inequalities

For states with a local hidden variable model (Mermin 1990) :

$$\left\langle X^{(1)}X^{(2)}X^{(3)}\right\rangle - \left\langle Y^{(1)}Y^{(2)}X^{(3)}\right\rangle - \left\langle X^{(1)}Y^{(2)}Y^{(3)}\right\rangle - \left\langle Y^{(1)}X^{(2)}Y^{(3)}\right\rangle \le 2.$$

For the GHZ state we have a Greenberger-Horne-Zeilinger-type violation of local realism:

$$\left\langle X^{(1)}X^{(2)}X^{(3)}\right\rangle = +1, \quad \left\langle Y^{(1)}X^{(2)}Y^{(3)}\right\rangle = -1,$$
$$\left\langle Y^{(1)}Y^{(2)}X^{(3)}\right\rangle = -1, \quad \left\langle X^{(1)}Y^{(2)}Y^{(3)}\right\rangle = -1.$$

For separable quantum states:

$$\left\langle X^{(1)}X^{(2)}X^{(3)}\right\rangle - \left\langle Y^{(1)}Y^{(2)}X^{(3)}\right\rangle \le 1.$$

There is not a separable quantum state for which

$$\left\langle X^{(1)}X^{(2)}X^{(3)}\right\rangle = +1,$$
$$\left\langle Y^{(1)}Y^{(2)}X^{(3)}\right\rangle = -1.$$

Criteria for cluster states

Let us construct an entanglement criterion with the stabilizing operators of cluster states.

Criterion 2: For fully separable states

$$\left\langle S_k^{(C_N)} \right\rangle + \left\langle S_{k+1}^{(C_N)} \right\rangle \le 1.$$

Proof. For product states, using the Cauchy-Schwarz ineq.

$$\left\langle S_k^{(GHZ_N)} \right\rangle + \left\langle S_{k+1}^{(GHZ_N)} \right\rangle = \left\langle Z^{(k)} X^{(k+1)} Z^{(k+2)} \right\rangle + \left\langle Z^{(k+1)} X^{(k+2)} Z^{(k+3)} \right\rangle =$$

$$\left\langle Z^{(k)} \right\rangle \left\langle X^{(k+1)} \right\rangle \left\langle Z^{(k+2)} \right\rangle + \left\langle Z^{(k+1)} \right\rangle \left\langle X^{(k+2)} \right\rangle \left\langle Z^{(k+3)} \right\rangle \le$$

$$\left| \left\langle X^{(k+1)} \right\rangle \right| \left| \left\langle X^{(k+2)} \right\rangle \right| + \left| \left\langle Z^{(k+1)} \right\rangle \right| \left| \left\langle X^{(k+2)} \right\rangle \right| \le 1.$$

It is easy to see that this is also true for mixed separable states.

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Measuring the fidelity

How to measure the fidelity with respect to a GHZ state, i.e., how to measure the operator

$$|GHZ_N\rangle\langle GHZ_N|?$$

In a typical experiment *only local measurements are possible*, thus it has to be decomposed into the sum of locally measurable terms. For three qubits we have:

$$|GHZ_3\rangle\langle GHZ_3| = \frac{1}{8}(1 + Z^{(1)}Z^{(2)} + Z^{(2)}Z^{(3)} + Z^{(1)}Z^{(3)} - 2X^{(1)}X^{(2)}X^{(3)})$$
$$+\frac{1}{16}\left(X^{(1)} + Y^{(1)}\right)\left(X^{(2)} + Y^{(2)}\right)\left(X^{(3)} + Y^{(3)}\right) + \frac{1}{16}\left(X^{(1)} - Y^{(1)}\right)\left(X^{(2)} - Y^{(2)}\right)\left(X^{(3)} - Y^{(3)}\right)$$

Problem: the number of local terms increases exponentially with the number of qubits.

O. Gühne and P. Hyllus, quant-ph.

Measurement settings

Measuring a local setting

$$\left\{O^{(1)}, O^{(2)}, O^{(3)}, \ldots, O^{(N)}\right\}$$

means measuring $O^{(k)}$ at qubits k=1,2,3,...,N several times. After the measurement outcomes are collected, all two, three-qubit, etc., correlations of the form

$$\left\langle O^{(k)} O^{(l)} \right\rangle, \left\langle O^{(k)} O^{(l)} O^{(m)} \right\rangle, \ldots$$

can be obtained.

Thus from the point of view of the measurement effort the number of local settings matters and not the number of correlations terms.

Lower bound on fidelity

Now we look for an operator which gives a lower bound on the fidelity but needs fewer measurements.

1. That is, we require that we have always a lower bound

$$P \leq \left| GHZ_N \right\rangle \left\langle GHZ_N \right|.$$

2. We look for this operator in the form (this is our *ansatz*)

$$P = \sum_k c_k \tilde{S}_k^{(GHZ_N)},$$

where c_k are constant.

3. We also require that *P* can be measured with the minimal two local measurement settings.

4. Under the above constraints, we want our lower bound to the highest possible for GHZ states + white noise.

Lower bounds on the fidelity

Results of the optimization:

1. Lower bound on the fidelity with respect to the GHZ state can be obtained from

$$P^{(GHZ_N)} = \frac{1+S_1^{(GHZ_N)}}{2} + \prod_{k>1} \frac{1+S_k^{(GHZ_N)}}{2} - 1.$$

1. z z z z z z z z ...

2. x x x x x x x x ...

2. Similar results for the cluster state

$$P^{(C_N)} = \prod_{k\ \mathrm{even}} \frac{1+S_k^{(C_N)}}{2} + \prod_{k\ \mathrm{odd}} \frac{1+S_k^{(C_N)}}{2} - 1.$$

1. x z x z x z x ...

2. z x z x z x z ...

G. Tóth and O. Gühne, PRL **94**, 060501 (2005).

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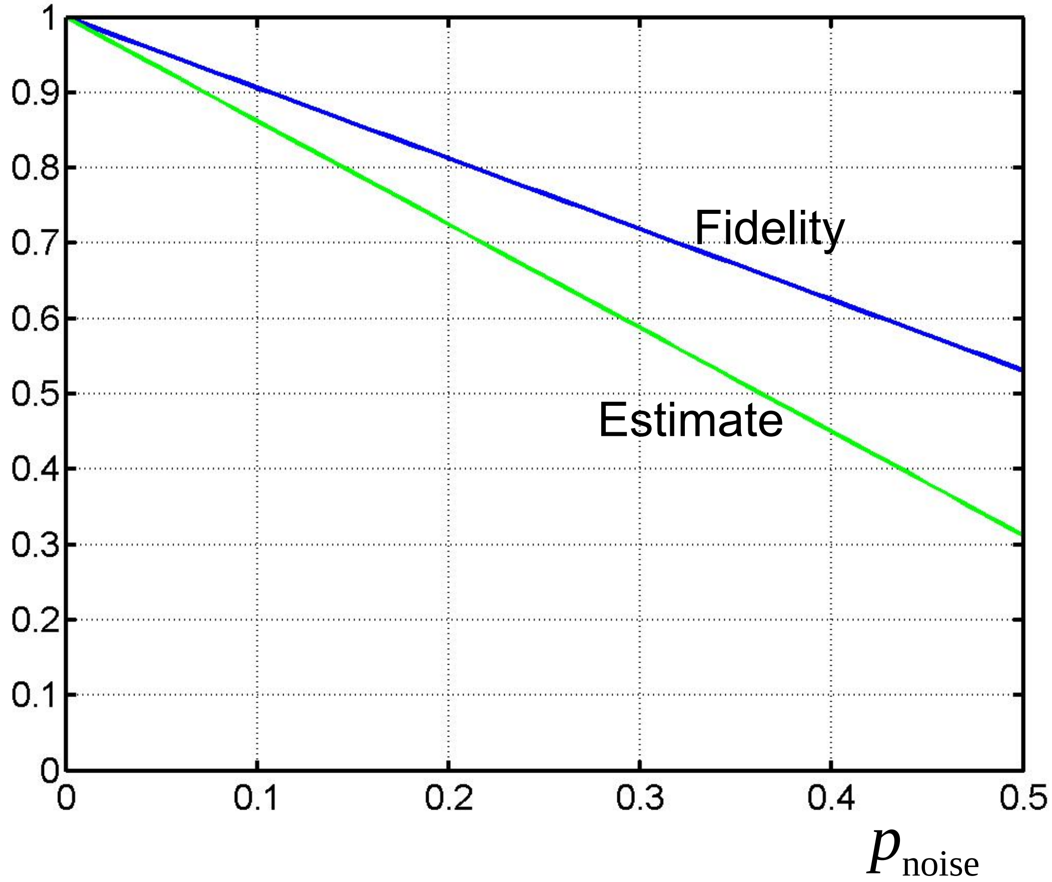
How good is our fidelity estimate?

Let us compare the fidelity and our estimate for noisy GHZ states

$$\rho(p_{\text{noise}}) = p_{\text{noise}}\frac{1}{2^4} + (1 - p_{\text{noise}})|GHZ_4\rangle\langle GHZ_4|.$$



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Genuine multi-qubit entanglement

Genuine three-qubit entanglement

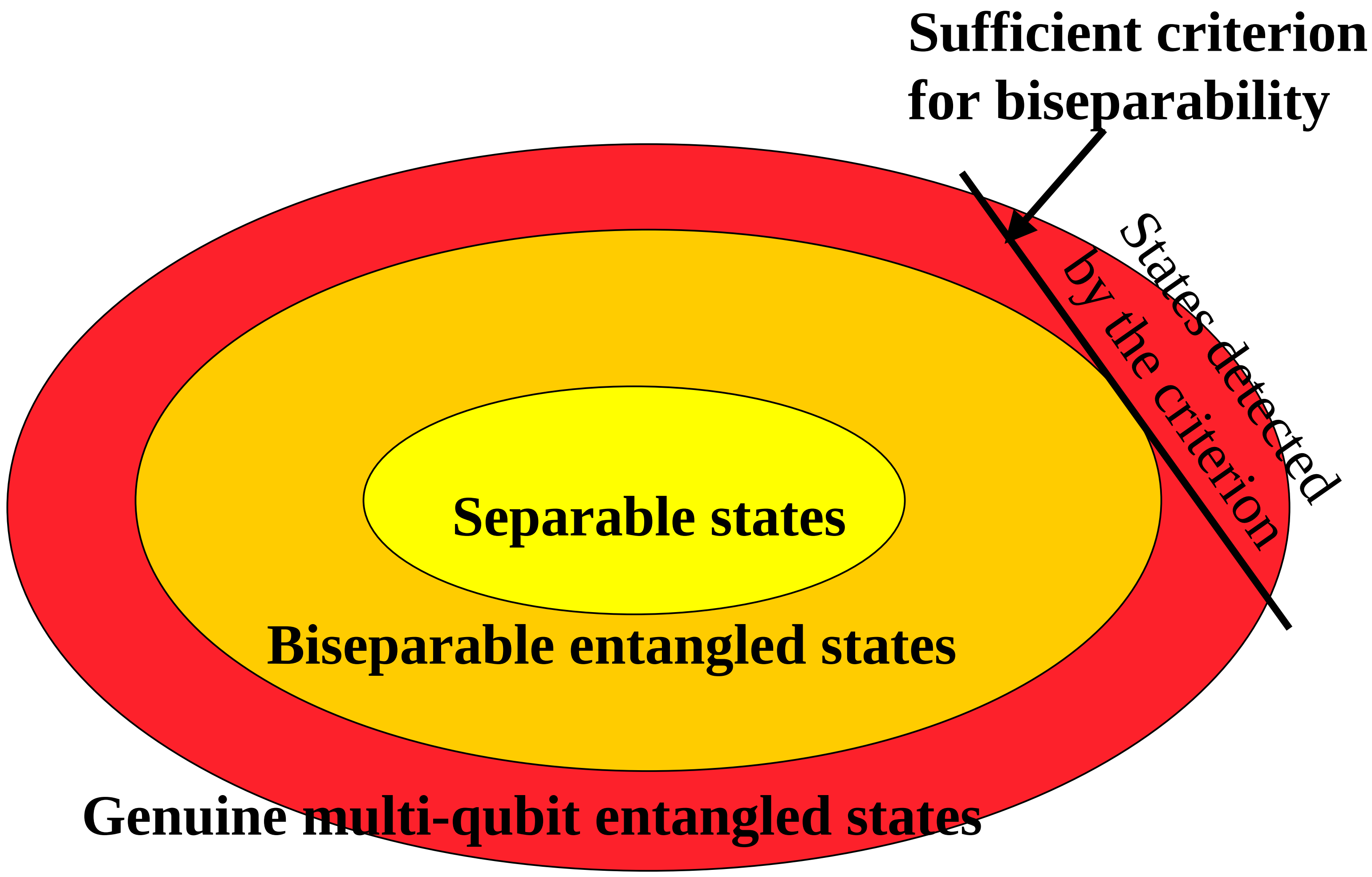
$$|000\rangle + |111\rangle$$

Biseparable entanglement

$$|001\rangle + |111\rangle = \left(|00\rangle + |11\rangle\right)|1\rangle$$

A mixed entangled state is biseparable if it is the mixture of biseparabe states (of possibly different partitions).

Genuine multi-qubit entanglement



Usual entanglement conditions

For states without N-qubit entanglement we have

$$\langle |GHZ_N\rangle\langle GHZ_N| \rangle \le \frac{1}{2},$$

Sacket *et. al.*, Nature **404**, 256 (2000).

$$\langle |C_N\rangle\langle C_N| \rangle \le \frac{1}{2}.$$

G. Tóth and O. Gühne, PRL **94**, 060501 (2005).

If one of these is violated then the state is *N*-qubit entangled.

Problem: too many local measurements are needed for the projector.

Our entanglement criteria

1. Criterion for detecting genuine *N*-qubit entanglement close to GHZ states

$$\left\langle \frac{1+S_1^{(GHZ_N)}}{2} + \prod_{k>1} \frac{1+S_k^{(GHZ_N)}}{2} \right\rangle \leq \frac{3}{2}$$

2. Criterion for detecting genuine *N*-qubit entanglement close to cluster states

$$\left\langle \prod_{k\ even} \frac{1+S_k^{(C_N)}}{2} + \prod_{k\ odd} \frac{1+S_k^{(C_N)}}{2} \right\rangle \leq \frac{3}{2}$$

They need only two measurement settings.

G. Tóth and O. Gühne, PRL. **94**, 060501 (2005).

Robustness to noise

Let us see again the noisy GHZ state

$$\rho(p_{\text{noise}}) = p_{\text{noise}} \frac{1}{2^N} + (1 - p_{\text{noise}}) \left| GHZ_N \right\rangle \left\langle GHZ_N \right|.$$

1. Our criterion detects it as N-qubit entangled if

$$p_{\text{noise}} \leq \frac{1}{3}.$$

2. Our other criterion for the cluster state detects the noisy cluster state as entangled if

$$p_{\text{noise}} \leq \frac{1}{4}.$$

Conclusions

We have discussed how to

- detect entanglement,
- estimate the fidelity with respect to a highly entangled state, and
- detect genuine multi-qubit entanglement

using the stabilizer theory in systems of many qubits.

For further information please see

G. Tóth and O. Gühne, PRL **94**, 060501 (2005).

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