Detecting Genuine Multipartite Entanglement with Only Two Measurement Settings

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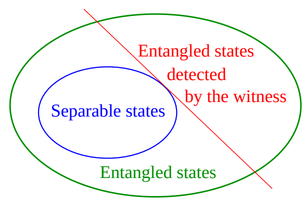
Introduction

- With the rapid development of quantum control it is now possible to study experimentally the entanglement of many qubits. In these experiments it is not sufficient to claim that "the state is entangled". A multi-qubit experiment presents something qualitatively new only if provably more than two qubits are entangled.
- Usual methods for detecting *genuine* multi-qubit entanglement need an experimental time growing *exponentially* with the number of qubits, making multi-qubit entanglement detection impossible even for modest size systems.
- We will show, it is still possible to decide whether a state is multi-qubit entangled without the need for exponentially growing resources, using only *local* measurements.
- We will present entanglement witnesses for this task [1]. Our constructions are robust against noise and require only two local measurement settings. They detect states close to GHZ and cluster states.

Entanglement Witnesses

- An operator $\mathcal{W}$ is an entanglement witness, if for every product state $\langle \mathcal{W} \rangle \geq 0$ and for *some* entangled states $\langle \mathcal{W} \rangle < 0$.
- Entanglement witnesses correspond to hyperplanes in the space of quantum states. States corresponding to a side of this plane are detected by the witness.



- We will construct witnesses which detect only genuine multi-party entanglement, i.e., they do not detect biseparable pure states or their mixtures. [For example, $(12)(3)$ biseparable pure states have the form $|\Psi\rangle = |\Psi_{12}\rangle \otimes |\Psi_3\rangle$. Note that qubit (3) is not entangled with the other qubits.]

Stabilizer Witnesses

- Entanglement witnesses are usually constructed using a projector [2] on a highly entangled state, $|\Psi\rangle$,
$$\mathcal{W}_P = c_P \cdot \mathbb{1} - |\Psi\rangle\langle\Psi|. \qquad (1)$$
These detect states in the proximity of an entangled state.
- We propose a different way of constructing witnesses for N-qubit quantum states
$$\mathcal{W} = c_0 \cdot \mathbb{1} - \sum_k c_k S_k, \qquad (2)$$
where c_k's are constants and S_k's stabilize [3] state $|\Psi\rangle$
$$S_k|\Psi\rangle = |\Psi\rangle. \qquad (3)$$
The S_k's are products of single-qubit operators.
- Advantage of our construction: it is easily measurable locally.

Stabilizing Operators

- Operators S_k form a commutative group called *stabilizer* [3]. Their theory plays a central role in error correction and fault tolerant computation.
- Stabilizing operators used for witnessing entanglement close to N-qubit GHZ state are
$$S_1^{(GHZ_N)} := \prod_{k=1}^{N} \sigma_x^{(k)},$$
$$S_k^{(GHZ_N)} := \sigma_z^{(k-1)}\sigma_z^{(k)};\ k \in \{2,3,..,N\}. \qquad (4)$$
- For detecting entanglement close to an N-qubit *cluster state* [4] we use the following stabilizing operators
$$S_1^{(C_N)} := \sigma_x^{(1)}\sigma_z^{(2)},$$
$$S_k^{(C_N)} := \sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}; k \in \{2,3,..,N-1\},$$
$$S_N^{(C_N)} := \sigma_z^{(N-1)}\sigma_x^{(N)}. \qquad (5)$$

Main Results

- Witness detecting *genuine* N-qubit entanglement around a GHZ state
$$\mathcal{W}_{GHZ_N} := 3 \cdot \mathbb{1} - 2\left[\frac{S_1^{(GHZ_N)} + \mathbb{1}}{2} + \prod_{k=2}^{N} \frac{S_k^{(GHZ_N)} + \mathbb{1}}{2}\right].$$
- The form of $\mathcal{W}_{GHZ_N}$ can be intuitively understood as follows. The first term in the square bracket is a projector on the subspace where $\langle S_1^{(GHZ_N)} \rangle = +1$. The second one is a projector on subspace where $\langle S_k^{(GHZ_N)} \rangle = +1$ for all $k \in \{2,3,..,N\}$. Clearly only a GHZ state gives $+1$ for both projectors.
- Witness detecting N-qubit entanglement around a cluster state [3]
$$\mathcal{W}_{C_N} := 3 \cdot \mathbb{1} - 2\left[\prod_{\text{even } k} \frac{S_k^{(C_N)} + \mathbb{1}}{2} + \prod_{\text{odd } k} \frac{S_k^{(C_N)} + \mathbb{1}}{2}\right].$$

Measurement Settings

- A *measurement setting* is the basic unit of experimental effort when detecting entanglement with local measurements.
- Measuring a local setting $\{O^{(k)}\}_{k=1}^{N}$ consists of performing simultaneously the von Neumann measurements $O^{(k)}$ on the corresponding parties. After these measurements are repeated several times, the coincidence probabilities for the outcomes are collected. Given these probabilities it is possible to compute all two-point correlations $\langle O^{(k)}O^{(l)}\rangle$, three-point correlations $\langle O^{(k)}O^{(l)}O^{(m)}\rangle$, etc.
- Thus from the point of view of experimental effort the number of settings counts rather than the number of correlation terms measured.

Comparison to Other Methods

- Remarkably, our witnesses need only two measurement settings independent from the number of qubits as shown in Fig. 1(a) and (b).
- When using Bell inequalities for entanglement detection, the number of measurement settings increases exponentially with the number of qubits. This is shown in Fig. 1(c) for the case when entanglement is detected close to GHZ states.

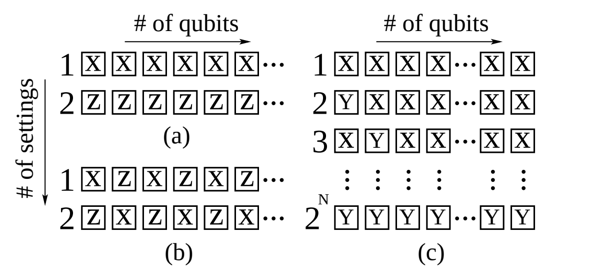


Figure 1. (a) The two measurement settings needed for our method for detecting entangled states close to GHZ and (b) cluster states. For each qubit the measured spin coordinate is indicated. (c) The 2^N settings required for Bell inequalities.

Noise Tolerance

- For practical purposes it is very important to know how large neighborhood of the GHZ and cluster states is detected by our witnesses. This is usually characterized by the robustness to noise.
- A three-qubit GHZ state after mixing with noise
$$\rho = (1-p)\rho_{GHZ_3} + p\frac{\mathbb{1}}{8} \qquad (6)$$
is detected as three-qubit entangled by the witness $\mathcal{W}_{GHZ_3}$ if $p < 0.4$. Thus our witness is robust against noise.
- A four-qubit cluster state is detected as four-qubit entangled by $\mathcal{W}_{C_4}$ if $p < 0.33$.
- For large number of qubits GHZ and cluster states are detected as multi-qubit entangled if $p < 0.33$ and $p < 0.25$, respectively.
- It can be proved that our witnesses are optimal from the point of view of noise tolerance. That is, there is no other witness requiring only two settings, which tolerates noise better.

Conclusions

- We have presented entanglement witnesses with simple local decomposition based on stabilizing operators. These witnesses detect genuine multi-party entanglement around GHZ and cluster states. They need only two measurement settings. The approach can straightforwardly be generalized for graph states. For more details see Reference [1].

Related bibliography:

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