Entanglement Witnesses with Simple Local Decomposition

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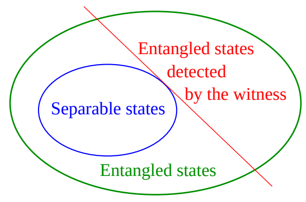
Introduction

- An operator W is an entanglement witness, if for every product state $\langle W\rangle \geq 0$ and for *some* entangled states $\langle W\rangle < 0$.
- Entanglement witnesses are usually constructed using a projector [1] to a highly entangled state, $|\Psi\rangle$,
$$W_P = c_P \cdot \mathbb{1} - |\Psi\rangle\langle\Psi|. \qquad (1)$$
- We propose a different way of constructing witnesses for N-qubit quantum states
$$W = c \cdot \mathbb{1} - \sum_{k=1}^{N} S_k, \qquad (2)$$
where S_k's stabilize [2] state $|\Psi\rangle$
$$S_k|\Psi\rangle = |\Psi\rangle. \qquad (3)$$
The S_k's are products of single-qubit operators. Advantage: it is much easier to measure the witnesses *locally*. However, the witness is more sensitive to noise than the projector-based witness.

Projector-based Witnesses

- The following entanglement witness detects *genuine* three-qubit entanglement based on a projector to a GHZ state, $|GHZ_3\rangle = |000\rangle + |111\rangle$,
$$W_{GHZ}^{P} := \frac{1}{2}\mathbb{1} - |GHZ_3\rangle\langle GHZ_3|. \qquad (4)$$
Witnesses similar to W_{GHZ}^{P} have already been used for experimental detection of entanglement [1].



- Witness W_{GHZ}^{P} detects only genuine three-party entanglement, i.e., it does not detect biseparable pure states or their mixtures. [For example, $(12)(3)$ biseparable pure states have the form $|\Psi\rangle = |\Psi_{12}\rangle \otimes |\Psi_3\rangle$. Note that qubit (3) is not entangled with the other qubits.]

Local Decomposition

- For an experiment, witness (4) must be decomposed into locally measurable terms [1]. The local decompostion of a projector-based witness is quite complicated. For example,
$$\begin{aligned} W_{GHZ}^{P} = & \\ & \frac{1}{8}\Big[3\cdot\mathbb{1} - \sigma_z^{(1)}\sigma_z^{(2)} - \sigma_z^{(2)}\sigma_z^{(3)} \\ - & \sigma_z^{(1)}\sigma_z^{(3)} - 2\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)} \\ + & \frac{1}{2}(\sigma_x^{(1)}+\sigma_y^{(1)})(\sigma_x^{(2)}+\sigma_y^{(2)})(\sigma_x^{(3)}+\sigma_y^{(3)}) \\ + & \frac{1}{2}(\sigma_x^{(1)}-\sigma_y^{(1)})(\sigma_x^{(2)}-\sigma_y^{(2)})(\sigma_x^{(3)}-\sigma_y^{(3)})\Big]. \end{aligned} \qquad (5)$$
- Decomposition (7) has 6 locally measurable terms. For these, 4 measurement settings are needed. (i.e., the $\sigma_z^{(1)}\sigma_z^{(2)}$ and $\sigma_z^{(2)}\sigma_z^{(3)}$ terms can be measured with one setup. One has to measure σ_z for each qubit and then compute the correlations.)

Stabilizer Witnesses

- The following witnesses are based on stabilizing operators. The first one detects *genuine* three-party entanglement around a GHZ state
$$W_{GHZ3} := 2\cdot\mathbb{1} - \sigma_z^{(1)}\sigma_z^{(2)} - \sigma_z^{(2)}\sigma_z^{(3)} - \sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}. \qquad (6)$$
It needs only 3 locally measurable terms and 2 measuring setups. (Compare with W_{GHZ}^{P} given in Eq. (5).)
- Witness detecting *genuine* four-qubit entanglement around a GHZ state
$$\begin{aligned} W_{GHZ4} := & \; 3\cdot\mathbb{1} - \sigma_z^{(1)}\sigma_z^{(2)} - \sigma_z^{(2)}\sigma_z^{(3)} - \sigma_z^{(3)}\sigma_z^{(4)} \\ - & \; \sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}\sigma_x^{(4)}. \end{aligned} \qquad (7)$$
- Witness detecting four-qubit entanglement around a *cluster* state [3]
$$\begin{aligned} W_{cl4} := & \; 3\cdot\mathbb{1} - \sigma_x^{(1)}\sigma_z^{(2)} - \sigma_z^{(1)}\sigma_x^{(2)}\sigma_z^{(3)} \\ - & \; \sigma_z^{(2)}\sigma_x^{(3)}\sigma_z^{(4)} - \sigma_z^{(3)}\sigma_x^{(4)}. \end{aligned} \qquad (8)$$

No Biseparability

- The witness W_{GHZ3} (see Eq. (6)) detects *genuine* three-party entanglement. *Proof.* First let us consider $(1)(23)$ biseparable product states. Then
$$\begin{aligned} & \left\langle\sigma_z^{(1)}\sigma_z^{(2)}\right\rangle + \left\langle\sigma_z^{(2)}\sigma_z^{(3)}\right\rangle + \left\langle\sigma_x^{(1)}\sigma_x^{(2)}\sigma_x^{(3)}\right\rangle \\ = & \left\langle\sigma_z^{(1)}\right\rangle\left\langle\sigma_z^{(2)}\right\rangle + \langle 1\rangle\left\langle\sigma_z^{(2)}\sigma_z^{(3)}\right\rangle \\ + & \left\langle\sigma_x^{(1)}\right\rangle\left\langle\sigma_x^{(2)}\sigma_x^{(3)}\right\rangle = \sum_{k=1}^{3} a_k b_k \leq |\vec{a}|\,|\vec{b}| = 2, \end{aligned} \qquad (9)$$
where for the Cauchy-Schwarz inequality
$$\begin{aligned} \vec{a} &= \left(\left\langle\sigma_z^{(1)}\right\rangle, \langle 1\rangle, \left\langle\sigma_x^{(1)}\right\rangle\right), \\ \vec{b} &= \left(\left\langle\sigma_z^{(2)}\right\rangle, \left\langle\sigma_z^{(2)}\sigma_z^{(3)}\right\rangle, \left\langle\sigma_x^{(2)}\sigma_x^{(3)}\right\rangle\right), \end{aligned}$$
and we used that $|\vec{a}| \leq \sqrt{2}$ and $|\vec{b}| \leq \sqrt{2}$. From Eq. (9) it follows, that for any $(1)(23)$ biseparable product state $\langle W_{GHZ3}\rangle \geq 0$. Similar proofs can be constructed for the $(2)(13)$ and $(12)(3)$ partitions.

Bounds for Separable States

- The following entanglement witness detects entanglement (but not necessarily genuine N-qubit entanglement) close to an N-qubit GHZ state
$$Q_{GHZN} := (N-1)\left(\mathbb{1} - \prod_{k=1}^{N}\sigma_x^{(k)}\right) - \sum_{k=1}^{N-1}\sigma_z^{(k)}\sigma_z^{(k+1)}. \qquad (10)$$
- The following entanglement witness detects entanglement for states close to an N-qubit cluster state [3,4]
$$Q_{clN} := \frac{N}{2}\cdot\mathbb{1} - \sum_{k=1}^{N}\sigma_z^{(k-1)}\sigma_x^{(k)}\sigma_z^{(k+1)}, \qquad (11)$$
where N is even and $\sigma_z^{(0)} = \sigma_z^{(N+1)} = 1$.
- For both stabilizer witnesses the number of terms increases *linearly* with the number of qubits.

Sensitivity to Noise

- A three-qubit GHZ state is detected as three-qubit entangled by the witness W_{GHZ3} after mixing with the totally mixed state
$$\rho = p\rho_{GHZ3} + (1-p)\frac{\mathbb{1}}{8} \qquad (12)$$
if $p > 2/3 \approx 0.67$. W_{GHZ3} is somewhat more sensitive to noise than the projector-based witness W_{GHZ}^{P} which detects entanglement if $p > 3/7 \approx 0.43$.
- A four-qubit GHZ state is detected as four-qubit entangled by W_{GHZ4} if $p > 3/4$.
- Many-qubit GHZ and cluster states are detected as entangled by Q_{GHZN} and Q_{clN}, respectively, if $p > 1/2$.

The Hamiltonian as a Witness

- Based on similar ideas, spin-chain Hamiltonians can also be used as entanglement witnesses. For example, let us consider the Heisenberg-chain Hamiltonian
$$H_H = \sum_{k=1}^{N-1}\sigma_x^{(k)}\sigma_x^{(k+1)} + \sigma_y^{(k)}\sigma_y^{(k+1)} + \sigma_z^{(k)}\sigma_z^{(k+1)}. \qquad (13)$$
- The expectation value for separable states is bounded by $E_{min,sep} = -N+1$. The proof is based on
$$\begin{aligned} & \left\langle\sigma_x^{(k)}\right\rangle\left\langle\sigma_x^{(k+1)}\right\rangle + \left\langle\sigma_y^{(k)}\right\rangle\left\langle\sigma_y^{(k+1)}\right\rangle \\ + & \left\langle\sigma_z^{(k)}\right\rangle\left\langle\sigma_z^{(k+1)}\right\rangle \leq 1. \end{aligned} \qquad (14)$$
- If the measured energy is less than $E_{min,sep}$ then the system is necessarily entangled. (The real ground state energy is around $-3N/2$.)

Conclusions

- We have presented a method for constructing entanglement witnesses with simple local decomposition based on stabilizing operators. These witnesses can detect genuine multi-party entanglement around GHZ and cluster states. The approach can straightforwardly be generalized for graph states.

Related bibliography:

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