Semidefinite programming in quantum information theory

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Linear programming

- Basic task: minimize a linear function of a vector $\vec{x}$ under linear constraints on the elements of $\vec{x}$.
- If there is a solution, it can be solved exactly.

Semidefinite programming

- Similar, but with semidefinite constraint.
- If there is a solution, it can always be solved exactly.

Outline

1. **Introduction**
   - Basic ideas
   - Solvable vs. not solvable by SDP
2. **The separability problem**
   - Separable states
   - PPT criterion

Basic ideas II

- In quantum physics, the density matrix ϱ is a positive semidefinite matrix
$$\varrho \geq 0.$$
- Its trace is one
$$\mathrm{Tr}(\varrho) = 1$$
and it is Hermitian
$$\varrho = \varrho^\dagger.$$
These conditions can easily be included in a semidefinite program.
- When we measure an operator X, the expectation value is
$$\langle X \rangle = \mathrm{Tr}(\varrho X).$$

Basic ideas III

- Let us see a simple example. We look for the minimum of

$$\langle X \rangle = \mathrm{Tr}(\varrho X)$$

  with the condition

$$\langle Y_n \rangle = \mathrm{Tr}(\varrho Y_n) = y_n$$

  for $n = 1, 2, .., N$, where X, Y_n are operators.

- We optimize over ϱ density matrices.

- This is again doable with semidefinite programming, although, there are better ways to do it.

Basic ideas III

- Program with MATLAB/YALMIP/MOSEK:

```
X=[0 1;1 0];
Y1=[1 0;0 -1];

rho=sdpvar(2,2,'hermitian','complex')

F=[rho>=0]+[trace(rho)==1]+[trace(rho*Y1)==0.2];

diagnostic=solvesdp(F,trace(X*rho));

minX=double(trace(X*rho))
```

- Result:

```
minX =
   -0.9798
```

N representability problem I

- Find ϱ of N qudits such that some reduced states are given.

  A. J. Coleman, Rev. Mod. Phys. 35, 668 (1963),

  for a summary of the literature see in Doherty, Parillo, Spedalieri, PRA 2005.

- Note that if only single-particle reduced states are given, we always have such a ϱ.

- If multiparticle reduced states are given, we do not always have such a ϱ.

N representability problem II

- Concrete example: find a two-qubit state such that the reduced states are

$$\varrho_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{pmatrix},$$

and

$$\varrho_2 = \begin{pmatrix} 0.5 & 0.2 \\ 0.2 & 0.5 \end{pmatrix}.$$

- The answer: the SDP finds such a state.

N representability problem III

- Program with MATLAB/YALMIP/MOSEK:

```
% 2 qubits
rho=sdpvar(4,4,'hermitian','complex')

% Reduced states
rho1=[0.5 0.1;0.1 0.5];
rho2=[0.5 0.2;0.2 0.5];

F=[rho>=0]+[trace(rho)==1];

% reduced states using an external routine
F=F+[keep_nonorm(rho,1)==rho1];
F=F+[keep_nonorm(rho,2)==rho2];

diagnostic=solvesdp(F,0);
is_there_a_problem=diagnostic.problem
rho_solution=double(rho)
```

N representability problem III

- Result

```
is_there_a_problem =
     0
rho_solution =
    0.2500    0.0500    0.1000    0.0344
    0.0500    0.2500    0.0344    0.1000
    0.1000    0.0344    0.2500    0.0500
    0.0344    0.1000    0.0500    0.2500
```

N representability problem IV

- Concrete example: find a two-qubit state such that the reduced states are
$$\varrho_1 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},$$
and
$$\varrho_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

- Note that the states are pure states and they are the eigestates of σ_x with an eigenvalue $+1$.

- The answer: the SDP finds such a state, it will find $\varrho_1 \otimes \varrho_2$.

N representability problem V

- Program with MATLAB/YALMIP/MOSEK:

```
% 2 qubits
rho=sdpvar(4,4,'hermitian','complex')

% Reduced states
rho1=[0.5 0.5;0.5 0.5];
rho2=[0.5 0.5;0.5 0.5];

F=[rho>=0]+[trace(rho)==1];

% reduced states using an external routine
F=F+[keep_nonorm(rho,1)==rho1];
F=F+[keep_nonorm(rho,2)==rho2];

diagnostic=solvesdp(F,0);
is_there_a_problem=diagnostic.problem
rho_solution=double(rho)
```

N representability problem VI

- Result

```
is_there_a_problem =
     0
rho_solution =
    0.2500    0.2500    0.2500    0.2500
    0.2500    0.2500    0.2500    0.2500
    0.2500    0.2500    0.2500    0.2500
    0.2500    0.2500    0.2500    0.2500
```

Nonlinear optimization

- Let us see a simple example. We look for the minimum of

$$\langle X_1 \rangle^2 + \langle X_2 \rangle^2 = \mathrm{Tr}(\varrho X_1)^2 + \mathrm{Tr}(\varrho X_2)^2$$

  with the condition

$$\langle Y_n \rangle = \mathrm{Tr}(\varrho Y_n) = y_n$$

  for $n = 1, 2, .., N$.

- We optimize over ϱ density matrices.

- This is again doable with semidefinite programming, minimizing $t_1 + t_2$ using the constraints

$$\begin{pmatrix} t_k & \mathrm{Tr}(\varrho X_k) \\ \mathrm{Tr}(\varrho X_k) & 1 \end{pmatrix} \geq 0$$

  for $k = 1, 2$.

Nonlinear optimization II

- Concrete example: we minimize

$$\langle\sigma_x\rangle^2 + \langle\sigma_z\rangle^2$$

  with the constraint

$$\langle Y_1\rangle = 0.2,$$

  where

$$Y_1 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

- The answer: we find a density matrix that corresponds to the minimum. For that state, we have

$$\langle\sigma_x\rangle^2 + \langle\sigma_z\rangle^2 = 0.01.$$

Nonlinear optimization III

- Program with MATLAB/YALMIP/MOSEK:

```
X1=[0 1;1 0]; X2=[1 0;0 -1]; Y1=[1 i;-i -1];

rho=sdpvar(2,2,'hermitian','complex')
t=sdpvar(1,1,'full','real')

F=[rho>=0]+[trace(rho)==1];

F=F+[trace(Y1*rho)==0.2];

M=[t trace(X1*rho);trace(X1*rho) t];
F=F+[M>=0];

diagnostic=solvesdp(F,t);

is_there_a_problem=diagnostic.problem

rho_solution=double(rho)
```

Nonlinear optimization IV

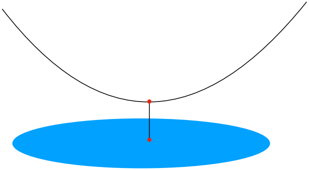
- Results:

```
is_there_a_problem =
     0
rho_solution =
   0.5500 + 0.0000i   0.0000 + 0.0500i
   0.0000 - 0.0500i   0.4500 + 0.0000i

>> trace(X2*rho_solution)
ans =
    0.1000

>> trace(X1*rho_solution)
ans =
     0
```

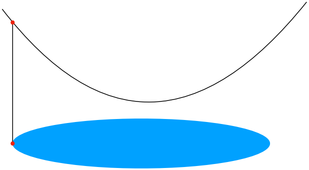
Outline

Solvable vs. not solvable by SDP

- Thus, we can minimize a convex function over the convex set of density matrices.



- However, we cannot maximize a function over the convex set of density matrices efficiently - the maximum is taken at the boundaries.



Outline

Mixed states: separable states vs. entangled states

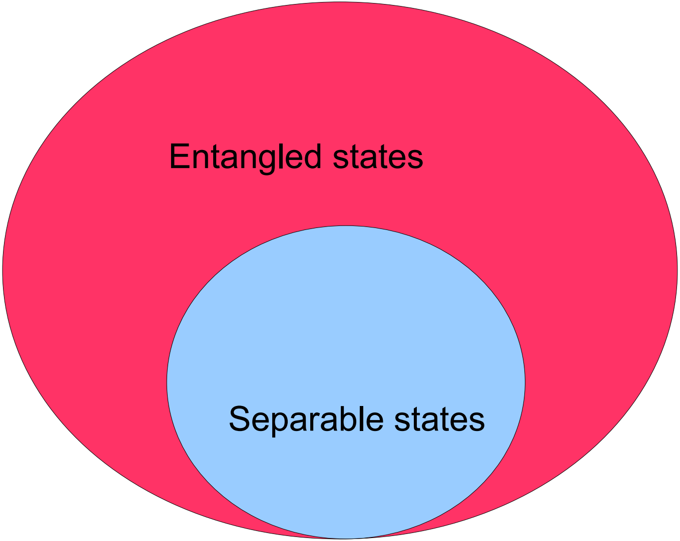
- For the mixed case, the definition of a separable state is (Werner 1989)
$$\rho_{\text{sep}} = \sum_k p_k [\rho_k^{(1)}]_A \otimes [\rho_k^{(2)}]_B.$$
A state that is not separable, is entangled.

- It is not possible to create entangled states from separable states, with LOCC.

- From many copies of two-qubit mixed entangled states, we can always distill a singlet using Local Operations and Classical Communication (LOCC).

- This is not true for higher dimensional systems. Not all quantum states are distillable.

Convex sets



Bipartite systems I

- Naive question: can we decide whether a state is separable with SDP? No, because we would need a constraint of the type

$$\varrho = (\varrho_1)_A \otimes (\varrho_2)_B.$$

- Alternatively, we would need a constraint for the reduced states of the *nth* subsytem

$$\mathrm{Tr}(\varrho^2_{\mathrm{red},n}) = 1.$$

Bipartite systems II

- How can we check separability using a brute force method? We can look for a separable decomposition with some $\rho_k^{(1)}, \rho_k^{(2)}$ numerically.

- Simpler problem, maximum for an operator expectation value for separable states

$$\max_{\rho_{\rm sep}} {\rm Tr}(X\rho_{\rm sep}) = \max_{\Psi_1,\Psi_2} \langle \Psi_1|\langle \Psi_2|X|\Psi_2\rangle\Psi_1\rangle.$$

- Numerically, we can try to find the maximum. In practice, we will find the maximum or something lower.

Outline

1. Introduction
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   - PPT criterion

The positivity of the partial transpose (PPT) criterion

Definition

For a separable state ϱ living in AB, the partial transpose is always positive semidefinite

$$\varrho^{TA} \geq 0.$$

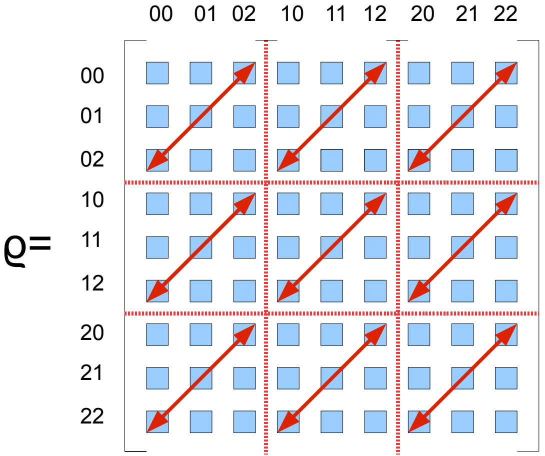
If a state does not have a positive semidefinite partial transpose, then it is entangled. A. Peres, PRL 1996; Horodecki *et al.*, PLA 1997.

- Partial transpose means transposing according to one of the two subsystems.
- For separable states

$$(T \otimes \mathbb{1})\varrho = \varrho^{TA} = \sum_k p_k (\varrho_k^{(1)})^T \otimes \varrho_k^{(2)} \geq 0.$$

The positivity of the partial transpose (PPT) criterion II

- How to obtain the partial transpose of a general density matrix? Example: 3×3 case.



The positivity of the partial transpose (PPT) criterion III

- If the relation

$$\varrho^{TA} \geq 0$$

  is violated then the state is entangled!

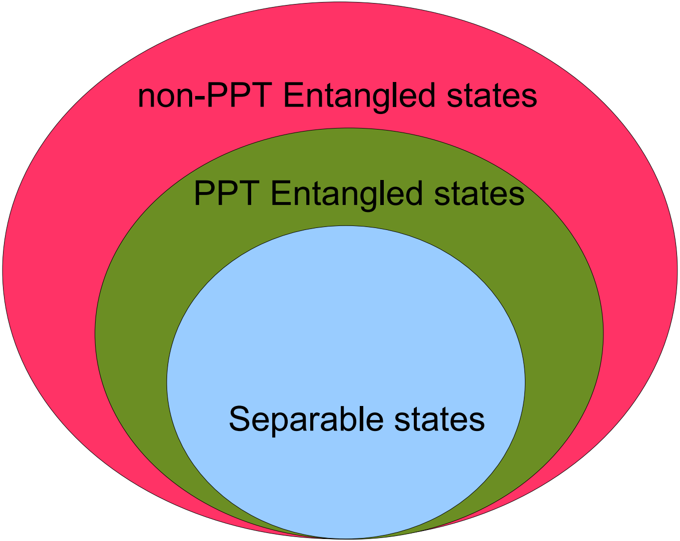
- For 2×2 and 2×3 systems it detects all entangled states.

- For larger systems, there are entangled states for which

$$\varrho^{TA} \geq 0.$$

  hold. They are bound entangled, not distillable.

Convex sets


non-PPT Entangled states
PPT Entangled states
Separable states

The positivity of the partial transpose (PPT) criterion IV

- Semidefinite programming can be used to optimize over PPT states.
- Find the minimum of an operator expectation value for PPT states:

  Minimize

  $$\langle X \rangle_{\varrho} \equiv \mathrm{Tr}(X\varrho)$$

  such that

  $$\begin{aligned} \varrho &= \varrho^{\dagger}, \\ \varrho &\geq 0, \\ \varrho^{TA} &\geq 0, \\ \mathrm{Tr}(\varrho) &= 1. \end{aligned}$$

The positivity of the partial transpose (PPT) criterion V

- Concrete example: look for the minimum of

$$\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \rangle$$

  for PPT states.

- The answer: the minimum is -1.

The positivity of the partial transpose (PPT) criterion VI

- Program with MATLAB/YALMIP/MOSEK:

```
sigmax=[0 1;1 0];sigmay=[0 -i;i 0];

% We want to minimze <A>
A=kron(sigmax,sigmax)+kron(sigmay,sigmay);

% Two qubits
rho=sdpvar(4,4,'hermitian','complex')

F=[rho>=0]+[trace(rho)==1];

% Using an external partial transpose routine
F=F+[pt_nonorm(rho,1)>=0];

diagnostic=solvesdp(F,trace(A*rho));
minimum=double(trace(A*rho))
```

The positivity of the partial transpose (PPT) criterion VII

- Result:

```
minimum =
   -1.0000
```

The positivity of the partial transpose (PPT) criterion VIII

- Thus, we find that the mininum of

$$\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \rangle$$

  for PPT states is -1.

- This is like finding a lower bound on the minimum for separable states.

- In practice, we often find the minimum for separable states, as in the example above.

G. Tóth, W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, and H. Weinfurter, New J. Phys. 2009.

The positivity of the partial transpose (PPT) criterion IX

- We can ask: is there a PPT fulfilling certain constraints?

  Look for ϱ such that

$$
\begin{aligned}
\varrho &= \varrho^\dagger, \\
\varrho &\geq 0, \\
\varrho^{TA} &\geq 0, \\
\mathrm{Tr}(\varrho) &= 1, \\
\mathrm{Tr}(X_k \varrho) &= x_k \text{ for } k = 1, 2, .., K.
\end{aligned}
$$

- If there is not such a ϱ then the state fulfilling the constraints is not PPT, and it is entangled (or it is not physical).
- One can use this to detect entanglement in experiments.

The positivity of the partial transpose (PPT) criterion X

- Concrete example: is there a PPT state with

$$\langle \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y \rangle = 1.1?$$

- The answer: there is no such a PPT state.

The positivity of the partial transpose (PPT) criterion XI

- Program with MATLAB/YALMIP/MOSEK: CHANGE!!!

```
sigmax=[0 1;1 0];sigmay=[0 -i;i 0];
A=kron(sigmax,sigmax)+kron(sigmay,sigmay);

% Two qubits
rho=sdpvar(4,4,'hermitian','complex')

F=[rho>=0]+[trace(rho)==1];

% Using an external partial transpose routine
% Condition <A>=1.1
F=F+[pt_nonorm(rho,1)>=0]+[trace(A*rho)==1.1];

diagnostic=solvesdp(F,0);
is_there_a_problem=diagnostic.problem
```

The positivity of the partial transpose (PPT) criterion XII

- Result:

```
is_there_a_problem =
     1
```

- That is, there is not such a PPT quantum state.