Quantum metrology from a quantum information science perspective

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Outline

1. **Introduction**
   - Motivation
2. **Basics of Quantum metrology**
   - Parameter estimation
   - Metrology with spin squeezed states
   - Metrology with GHZ states
   - Spin squeezing and entanglement
3. **Quantum Fisher information**
   - The quantum Fisher information
   - Properties of the quantum Fisher information
4. **Quantum Fisher information and entanglement**
   - Entanglement and multipartite entanglement
   - Definition of macroscopic superposition
   - Speed of the quantum evolution
5. **The effect of noise**
6. **Conclusions**

Quantum Metrology

- Metrology with an ensemble of particles
- It is not possible to estimate a parameter with a zero variance, even in an ideal situation.
- The precision decreases with the number of particles N.
- Shot-noise scaling

$$(\Delta\theta)^2 \sim \frac{1}{N}.$$

- Heisenberg scaling

$$(\Delta\theta)^2 \sim \frac{1}{N^2}.$$

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The type of system we consider here

- System of many particles.
- The particles cannot be accessed individually.
- There is no interaction between the particles.

Collective observables

- For spin-$\frac{1}{2}$ particles. The collective quantities are

$$J_l := \sum_{n=1}^{N} j_l^{(n)}$$

for $l = x, y, z$, where $j_l^{(n)}$ are the components of the angular momentum of the n^{th} particle.

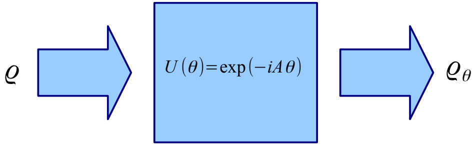
- A typical Hamiltonian: describing the action of a magnetic field pointing in the $\vec{b}$-direction

$$H_B = \gamma B J_{\vec{b}},$$

where γ is the gyromagnetic ratio, B is the strength of the magnetic field, $\vec{b}$ is the direction of the field, and $J_{\vec{b}}$ is the angular momentum component parallel with the field.

- No interaction terms, thus starting from a product state we arrive also at a product state.

Basic task of Quantum Metrology



- The Hamiltonian we discussed, with the choice of $\hbar = 1$, generates the dynamics

$$U_\theta = e^{-iJ_{\vec{n}}\theta},$$

where we defined the angle θ that depends on the evolution time t

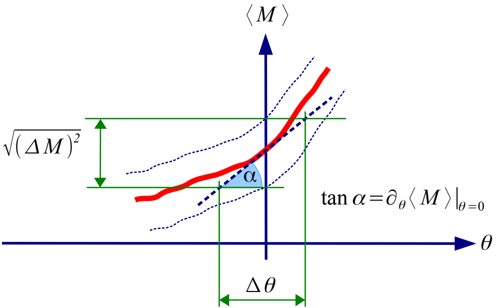
$$\theta = \gamma B t.$$

- A basic task in quantum metrology is to estimate the small parameter θ by measuring the expectation value of by M.
- If the evolution time t is a constant then estimating θ is equivalent to estimating the magnetic field B.

The error propagation formula

- We measure M and estimate the parameter θ.
- The variance of the parameter estimation is

$$(\Delta\theta)^2 = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2}.$$



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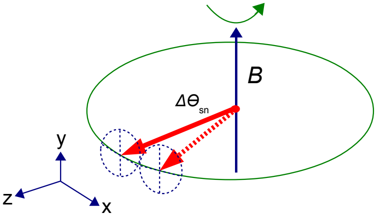
Metrology with a fully polarized ensemble

- With fully polarized states, the parameter estimation has a finite variance.



Metrology with a fully polarized ensemble II

- Let us measure

$$M = J_x.$$

- With this,

$$\langle M \rangle = \frac{N}{2}\cos(\theta), \quad (\Delta M)^2 = \frac{N}{4}\sin^2(\theta).$$

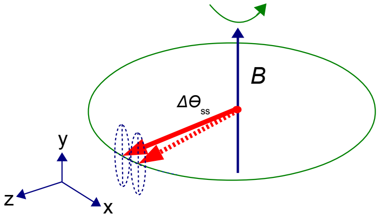
- The precision is

$$(\Delta\theta)^2|_{\theta=0} = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{N}.$$

- Note that $(\Delta J_x)^2 = \frac{N}{4}$.

Metrology with spin squeezed states

- Spin squeezing can decrease $(\Delta J_x)^2$. We can reach $(\Delta J_x)^2 < \frac{N}{4}$.
- This also decreases the variance of the parameter estimation.



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Metrology with GHZ states

- Let us take a GHZ state.
- Let us employ the dynamics

$$U = e^{-iJ_z\theta}.$$

- The dynamics of the state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|000..000\rangle + |111..111\rangle e^{-iN\theta}\right).$$

Metrology with GHZ states II

- Let us measure

$$M = \sigma_x^{\otimes N}.$$

- With this,

$$\langle M \rangle = \cos(N\theta), \quad (\Delta M)^2 = \sin^2(N\theta).$$

- The precision is

$$(\Delta\theta)^2|_{\theta=0} = \frac{(\Delta M)^2}{|\partial_\theta \langle M \rangle|^2} = \frac{1}{N^2}.$$

- Tested for 3 qubits.

[Nature 2001. (NIST, Boulder, Colorado).]

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Spin squeezing and entanglement

Spin squeezing inequality

$$\xi_s^2 := N \frac{(\Delta J_x)^2}{\langle J_y \rangle^2 + \langle J_z \rangle^2} \geq 1.$$

- If a state violates the above inequality, then it is entangled (i.e., not fully separable).
- In order to violate the inequality
  - its denominator must be large, and
  - its numerator must be small.
- Hence, the inequality detects states that have a large spin in some direction, while a small variance of a spin component in an orthogonal direction.

[A. Sørensen, L.M. Duan, J.I. Cirac and P. Zoller, Nature 63, 409 (2001).]

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The quantum Cramér-Rao bound

- We would like to estimate the parameter θ for the evolution $U = e^{iA\theta}$.

- The variance of the unbiased estimator $\hat{\theta}$ is lower bounded as

$$\mathrm{var}(\hat{\theta}) \geq \frac{1}{F_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information for the quantum state ϱ and Hamiltonian A.

Quantum Fisher information

- The quantum Fisher information F_Q can be computed easily with a closed formula. Let us assume that a density matrix is given in its eigenbasis as

$$\varrho = \sum_k \lambda_k |k\rangle\langle k|.$$

- The quantum Fisher information is given as

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2.$$

[C. Helstrom, Quantum Detection and Estimation Theory (1976);

A. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (1982);

S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 72 3439 (1994);
S.L. Braunstein, C.M. Caves, and G.J. Milburn, Ann. Phys., NY 247 135 (1996)]

- Family of the quantum Fisher informations

[D. Petz, J. Phys. A: Math. Gen 35, 929 (2002);
D. Petz, Quantum Information Theory and Quantum Statistics (Springer: Berlin, 2008).]

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Fundamental properties of the Quantum Fisher information

(i) For pure states, $F_Q[\varrho, A] = 4(\Delta A)^2$.

(ii) For all quantum states, it can be proven that

$$F_Q[\varrho, A] \le 4(\Delta A)^2.$$

This provides an easily computable upper bound on the quantum Fisher information.

(iii) The quantum Fisher information is convex in the state

$$F_Q[p\varrho_1 + (1-p)\varrho_2, A] \le pF_Q[\varrho_1, A] + (1-p)F_Q[\varrho_2, A].$$

Fundamental properties of the Quantum Fisher information II

(iv) Recently, it has turned out that the quantum Fisher information is the largest convex function that fulfills (i). This can be stated in a concise form as follows. Let us consider a very general decomposition of the density matrix

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|,$$

where $p_k > 0$ and $\sum_k p_k = 1$. The quantum Fisher information can be given as the convex roof of the variance,

$$F_Q[\varrho, A] = 4 \inf_{\{p_k, |\Psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\Psi_k},$$

where the optimization is over all the possible decompositions.
[G. Tóth G and D. Petz, Phys. Rev. A 87, 032324 (2013).]

Quantum Fisher information and variance

Analogously, it can also be proven that the concave roof of the variance is itself

$$(\Delta A)^2_{\varrho} = \sup_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k (\Delta A)^2_{\Psi_k}.$$

Summary: For any decomposition $\{p_k, |\Psi_k\rangle\}$ of the density matrix ϱ we have

$$\tfrac{1}{4} F_Q[\varrho, A] \leq \sum_k p_k (\Delta A)^2_{\Psi_k} \leq (\Delta A)^2_{\varrho},$$

where the upper and the lower bounds are both tight
[G. Tóth and D. Petz, Phys. Rev. A 87, 032324 (2013); S. Yu, arXiv:1302.5311 (2013).]

Further properties of the qFi

(i) The formula for F_Q does not depend on the diagonal elements $\langle i|A|i\rangle$. Hence,

$$F_Q[\varrho, A] = F_Q[\varrho, A + D],$$

where D is a matrix that is diagonal in the basis of the eigenvectors of ϱ, i.e., $[\varrho, D] = 0$.

(ii) The following identity holds for all unitary dynamics U

$$F_Q[U\varrho U^\dagger, A] = F_Q[\varrho, U^\dagger A U].$$

(iii) The quantum Fisher information is additive under tensoring

$$F_Q[\varrho^{(1)} \otimes \varrho^{(2)}, A^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes A^{(2)}] = F_Q[\varrho^{(1)}, A^{(1)}] + F_Q[\varrho^{(2)}, A^{(2)}].$$

Properties of the Quantum Fisher information II

(iv) The quantum Fisher information is additive under a direct sum

$$F_Q[\bigoplus_k p_k \varrho_k, \bigoplus_k A_k] = \sum_k p_k F_Q[\varrho_k, A_k],$$

(v) State mixed with white noise

$$F_Q[\varrho_{\rm noisy}(p), A] = \frac{p^2}{p + \frac{1-p}{2} d^{-N}} F_Q[|\Psi\rangle\langle\Psi|, A].$$

Symmetric logarithmic derivative

- Symmetric logarithmic derivative L

$$\frac{d\varrho_\theta}{d\theta} = \frac{1}{2}(L\varrho_\theta + \varrho_\theta L).$$

- von Neumann equation with the Hamiltonian A

$$\frac{d\varrho_\theta}{d\theta} = i(\varrho_\theta A - A\varrho_\theta).$$

- The right-hand sides of the two equations above must be equal. Hence,

$$L = 2i \sum_{i,j} \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l} |k\rangle\langle l| \langle k|A|l\rangle,$$

  where λ_k and $|k\rangle$ are the eigenvalues and eigenvectors, respectively, of the density matrix ϱ.

- Relation to the quantum Fisher information

$$F_Q[\varrho, A] = \mathrm{Tr}(\varrho L^2).$$

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Quantum Fisher information and entanglement

Bound for separable states

$$F_Q[\varrho, J_z] \leq N.$$

[L. Pezzé and A. Smerzi, Phys. Rev. Lett. (2007).]

- Hence, based on the Cramér-Rao bound we have

$$(\Delta\theta)^{-2}|_{\theta=0} \leq F_Q[\varrho, J_z] \leq N.$$

Bounds for separability and multipartite entanglement

Bound for k-producible separable states

For N-qubit k-producible states, the quantum Fisher information is bounded from above by

$$F_Q[\varrho, J_l] \leq sk^2 + (N - sk)^2,$$

where s is the integer part of $\frac{N}{k}$.
[G. Tóth, Phys. Rev. A (2012); P. Hyllus et al., Phys. Rev. A (2012).]

- For the case N divisible by k as

$$F_Q[\varrho, J_{\vec{n}}] = Nk.$$

- Thus, full entanglement is needed for maximal precision.

Bounds for separability and multipartite entanglement II

- For N-qubit k-producible states, the quantum Fisher information is bounded from above as

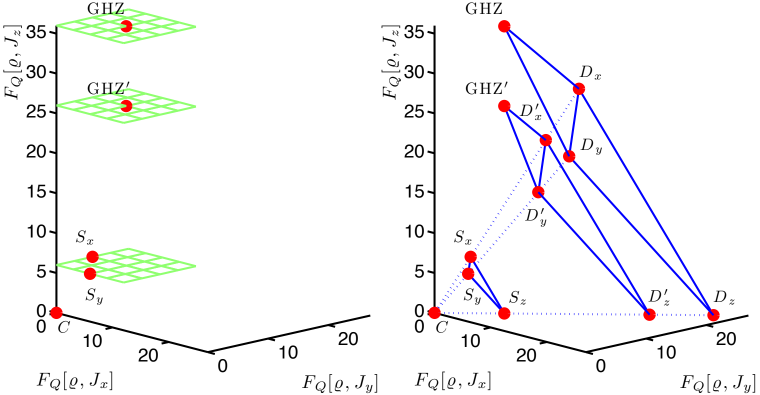
$$\mathrm{avg}_{\mathbf{n}} F_Q[\varrho, J_{\vec{n}}] \leq \begin{cases} \frac{1}{3}sk(k+2) + \frac{1}{3}(N-sk)(N-sk+2) & \text{if } N-sk \neq 1, \\ \frac{1}{3}sk(k+2) + \frac{2}{3} & \text{if } N-sk = 1, \end{cases}$$

  where s is the integer part of $\frac{N}{k}$.

- Thus, full entanglement is needed for maximal precision (e.g., GHZ state, Dicke state).

[Tóth, Phys. Rev. A (2012); P. Hyllus et al., Phys. Rev. A (2012).]

Bounds for separability and multipartite entanglement III



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Macroscopic superposition

- Define $\mathcal{A}_{\rm coll}$ as the set of collective operators given as

$$A_{\rm coll} = \sum_{n=1}^{N} a^{(n)}.$$

  Here $a^{(n)}$ are single-particle operators acting on the $n^{\rm th}$ particle and $||a^{(n)}|| = 1$. For spin-$\frac{1}{2}$ particles and traceless $a^{(n)}$, such operators are just the collective angular momentum component J_z apart from local unitaries and a constant factor.

- For pure states, define index p as

$$\max_{A\in\mathcal{A}_{\rm coll}} (\Delta A)^2 = \mathcal{O}(N^p).$$

  Here, $f(x) = \mathcal{O}(x^m)$ means that $\lim_{x\to\infty} \frac{f(x)}{x^m} = {\rm constant} > 0$.

- The index p is confined in a range $1 \le p \le 2$. For product states we have $p = 1$. The state is called macroscopically entangled if $p = 2$.

[A. Shimizu and T. Morimae, PhysRevLett 95, 090401 (2005).]

Macroscopic superposition II

- Extension to mixed states. Define

$$N_{\text{eff}}^{F}(\varrho) = \frac{1}{4N} \max_{A \in \mathcal{A}_{\text{coll}}} F_Q[\varrho, A].$$

The state is called macroscopically entangled if $N_{\text{eff}}^{F} = \mathcal{O}(N)$.

- If N divisible by k, for k-producible states we have

$$N_{\text{eff}}^{F}(\varrho) = k.$$

For a macroscopic superposition, the state must be k-particle entangled such that

$$k = \mathcal{O}(N).$$

We do not need full N-body entanglement.

[F. Fröwis and W. Dür, New J. Phys. 14 093039 (2012).]

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Speed of the quantum evolution and quantum Fisher information

- Fidelity between the initial state ϱ and the final state ϱ_θ. Well-known formula leads to

$$F_B(\varrho, \varrho_\theta) = 1 - \theta^2 \frac{F_Q[\varrho, A]}{4} + \mathcal{O}(\theta^3),$$

  where the parameter θ is small and the system Hamiltonian is given by A.

- Hence, for a fast evolution we need a large quantum Fisher information.

- If $A \in \mathcal{A}_{\rm coll}$ then this also means a large multipartite entanglement.

[A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Phys. Rev. Lett. 110, 050403 (2013); M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Phys. Rev. Lett. 110 050402 (2013).]

Effect of noise

- Simple example: Spin squeezed states, effect of uncorrelated noise. A particle with a state ϱ_1 passes trough a map

$$\epsilon_p(\varrho_1) = (1-p)\varrho_1 + p\frac{\mathbb{1}}{2}.$$

- The final state can be given as a mixture

$$\epsilon_p^{\otimes N}(\varrho) = \sum_{n=0}^{N} p_n \varrho_n,$$

where the state obtained after n particles decohered is

$$\varrho_n = \frac{1}{N!} \sum_k \Pi_k \left[\left(\frac{\mathbb{1}}{2}\right)^{\otimes n} \otimes \mathrm{Tr}_{1,2,\ldots,n}(\varrho) \right] \Pi_k^\dagger.$$

The summation is over all permutations Π_k. We took advantage of the permutational invariance.

Effect of noise II

- The quantity p_n in the decomposition is obtained with the binomial coefficients as

$$p_n = \binom{N}{n} p^n (1-p)^{(N-n)}.$$

- For the noisy state, the variance of the collective angular momentum component can be bounded from below as

$$(\Delta J_x)^2 \geq \sum_n p_n (\Delta J_x)^2_{\varrho_n} \geq \sum_n p_n \tfrac{n}{4} = \tfrac{pN}{4}.$$

- Hence, we get the shot-noise scaling for large N

$$(\Delta\theta)^{-2} \leq \frac{\frac{N^2}{4}}{\frac{pN}{4}} \propto N.$$

[G. Tóth and I. Apellaniz, J. Phys. A, in press, special issue "50 years of Bell's theorem"; arXiv:1405.4878.]

Effect of noise III

- In general, uncorrelated noise destroys the Heisenberg scaling for large N.
- However, correlated noise does not necessarily destroy the Heisenberg scaling.

[G. Tóth and I. Apellaniz, J. Phys. A, in press, special issue "50 years of Bell's theorem"; arXiv:1405.4878.]

Conclusions

- We discussed several aspects of quantum metrology.
- For the transparencies, see

www.gtoth.eu

- See also

G. Tóth and I. Apellaniz,
Quantum metrology from a quantum information science perspective,
J. Phys. A, in press, special issue "50 years of Bell's theorem";
arXiv:1405.4878.

THANK YOU FOR YOUR ATTENTION!

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