Collective Randomized Measurements in Quantum Information Processing

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The concept of randomized measurements on individual particles has proven to be useful for analyzing quantum systems and is central for methods like shadow tomography of quantum states. We introduce *collective* randomized measurements as a tool in quantum information processing. Our idea is to perform measurements of collective angular momentum on a quantum system and actively rotate the directions using simultaneous multilateral unitaries. Based on the moments of the resulting probability distribution, we propose systematic approaches to characterize quantum entanglement in a collective-reference-frame-independent manner. First, we show that existing spin-squeezing inequalities can be accessible in this scenario. Next, we present an entanglement criterion based on three-body correlations, going beyond spin-squeezing inequalities with two-body correlations. Finally, we apply our method to characterize entanglement between spatially separated two ensembles.

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Introduction—Rapid advances in quantum technology have made it possible to manipulate and control increasingly complex quantum systems. However, as the number of particles increases, the dimension of the Hilbert space grows exponentially, making it difficult to analyze quantum states fully. One way to address this issue is to rotate measurement directions with random unitaries and consider the moments of the resulting probability distribution. This method can provide essential quantum information about the system and give several advantages in characterizing quantum systems [1,2].

First, it allows us to obtain knowledge of the quantum state, reducing the experimental effort compared with the standard way of quantum state tomography. Second, it is useful when some prior information about the state is not available, such as when an experiment is intended to create a particular quantum state. Third, and most importantly, it does not need careful calibration and alignment of measurement directions or the sharing of a common frame of reference between the parties.

Several proposals have been put forward in the field of randomized measurements to detect bipartite [3–10] and multipartite entanglement [11–16]. Another research line of randomized measurements has estimated several useful functions of quantum states such as state's purity [17], Rényi entropies [3,4,18], state's fidelities [19], scrambling [20], many-body topological invariants [21,22], the von Neumann entropy [23], quantum Fisher information [24–26], and the moments of the partially transposed quantum state [27–32]. Also, in the framework of shadow tomography, the techniques of randomized measurements are used to predict future measurements via estimators in data collections [33–38].

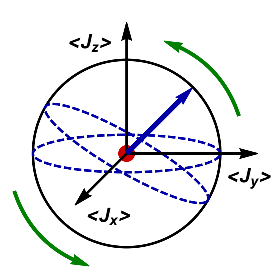

FIG. 1. Sketch of the collective Bloch sphere with the coordinates $(\langle J_x\rangle, \langle J_y\rangle, \langle J_z\rangle)$. Many-body spin singlet states are represented by a dot at the center (red), which does not change under any multilateral unitary transformations $U^{\otimes N}$ (green arrows). Spin measurement in the z direction is rotated randomly (blue arrow). This Letter proposes systematic methods to characterize spin-squeezing entanglement in an ensemble of particles by rotating a collective measurement direction randomly over this sphere.

Although many tools in randomized measurements were already presented, still the current findings are not fully comprehensive in several respects. One limitation of the results presented so far is the assumption that local subsystems can be controlled individually. However, this may not be available in an ensemble of quantum particles such as cold atoms [39], or trapped ions [40], or Bose-Einstein condensates with spin squeezing [41–43]. Such quantum systems can be characterized by measuring global quantities such as collective angular momenta [44–48].

Another practical challenge is that powerful entanglement detection requires many operational resources. For instance, Refs. [6,7] suggested that at least fourth-order moments of randomized measurements are needed to characterize a very weak form of entanglement, known as bound entanglement [49]. In fact, their practical implementation may require a significant amount of randomized measurement due to the limited availability of unitary designs [2,5,50,51].

In this Letter, we generalize the concept of randomized measurements on individual particles to the notion of *collective* randomized measurements. The main idea is to perform collective random rotations on a multiparticle quantum system before a fixed measurement and consider the moments of the resulting distribution of results with respect to the randomly chosen rotations. We will apply this idea to different scenarios and present several entanglement criteria in a collective-reference-frame-independent (CRFI) manner. Note that a similar idea of collective randomization has recently been used in the context of classical shadow tomography [52] to classify trivial and topologically ordered phases in many-body quantum systems.

We first show that spin-squeezing entanglement in permutationally symmetric N-particle systems can be characterized completely. Second, even in nonsymmetric cases, we demonstrate that the second-order moment can detect multiparticle bound entanglement. Third, we further introduce a criterion to certify multiparticle bound entanglement with antisymmetric correlations via third-order moments. Finally, we generalize the method to verify entanglement between spatially separated two quantum ensembles.

Collective randomized measurements—Consider a quantum ensemble that consists of N spin-$\frac{1}{2}$ particles in a state $\varrho \in \mathcal{H}_2^{\otimes N}$. Suppose that each particle in this ensemble cannot be controlled individually, and one can instead measure the collective angular momentum

$$J_l = \frac{1}{2}\sum_{i=1}^{N} \sigma_l^{(i)}, \tag{1}$$

with Pauli spin matrices $\sigma_l^{(i)}$ for $l = x, y, z$ acting on ith subsystem.

Let us perform measurements with J_z and rotate the collective direction in an arbitrary manner. We introduce an expectation value and its variance according to a random unitary,

$$\langle J_z \rangle_U = \mathrm{tr}[\varrho U^{\otimes N} J_z (U^\dagger)^{\otimes N}], \tag{2a}$$

$$(\Delta J_z)_U^2 = \langle J_z^2 \rangle_U - \langle J_z \rangle_U^2. \tag{2b}$$

These depend on the choice of collective simultaneous multilateral unitary operations $U^{\otimes N}$. Now we define a linear combination as

$$f_U(\varrho) = \alpha (\Delta J_z)_U^2 + \beta \langle J_z \rangle_U^2 + \gamma, \tag{3}$$

where α, β, γ are real constant parameters. The function $f_U(\varrho)$ can be determined experimentally by observing $\langle J_z \rangle_U$ and $(\Delta J_z)_U^2$ as each parameter can be adjusted in the postprocessing.

The key idea to detect entanglement in ϱ is to take a sample over collective local unitaries and consider the rth moments of the resulting distribution,

$$\mathcal{J}^{(r)}(\varrho) = \int dU [f_U(\varrho)]^r, \tag{4}$$

where the integral is taken according to the Haar measure. This collective unitary transformation can be written as $U^{\otimes N} = e^{i\boldsymbol{u}\cdot\boldsymbol{J}}$, where $\boldsymbol{u} = (u_x, u_y, u_z)$ is a three-dimensional unit vector and $\boldsymbol{J} = (J_x, J_y, J_z)$ is a vector of collective angular momenta. The randomization of Haar collective unitaries corresponds to the uniform randomization over the three-dimensional sphere in the coordinates $(\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$. This sphere is known as the *collective Bloch sphere* [46,47,53] in an analogy of the standard Bloch sphere in a single-qubit system, illustrated in Fig. 1.

It is essential that, by definition, the moments are *invariant* under any collective local unitary transformation,

$$\mathcal{J}^{(r)}[V^{\otimes N} \varrho (V^\dagger)^{\otimes N}] = \mathcal{J}^{(r)}(\varrho), \tag{5}$$

for a collective local unitary $V^{\otimes N}$ for 2×2 unitaries V. In the following, we will discuss CRFI entanglement detection based on the moments $\mathcal{J}^{(r)}$.

Permutationally symmetric states—To proceed, let us recall that an N-qubit state ϱ is called *permutationally symmetric (bosonic)* if it satisfies $P_{ab}\varrho = \varrho P_{ab} = \varrho$, for all $a, b \in \{1, 2, \ldots, N\}$ with $a \neq b$. Here, P_{ab} is an orthogonal projector onto the so-called symmetric subspace that remains invariant under all the permutations. Note that P_{ab} can be written as $P_{ab} = (\mathbb{1} + \mathbb{S}_{ab})/2$ with the SWAP (flip) operator $\mathbb{S}_{ab} = \sum_{i,j} |ij\rangle\langle ji|$ that can exchange qubits a, b: $\mathbb{S}_{ab}|\psi_a\rangle \otimes |\psi_b\rangle = |\psi_b\rangle \otimes |\psi_a\rangle$. We stress that the notion of permutational symmetry is stronger than permutational invariance, defined by $P_{ab}\varrho P_{ab} = \varrho$ [54].

There are many studies on the entanglement of permutationally symmetric states [54–60]. In general, a state ϱ is said to contain multipartite entanglement if it cannot be written as the fully separable state

$$\varrho_{\mathrm{fs}} = \sum_k p_k \left| a_k^{(1)}, a_k^{(2)} \cdots a_k^{(N)} \right\rangle \left\langle a_k^{(1)}, a_k^{(2)} \cdots a_k^{(N)} \right|, \qquad (6)$$

where the p_k form a probability distribution. Importantly, for any N-particle permutationally symmetric state the pure states in a decomposition like Eq. (6) need to be symmetric, too; so a symmetric state is either fully separable or genuinely multipartite entangled (GME) [55,59], where GME states cannot be written in any separable form for all bipartitions. One sufficient way to prove GME for a symmetric state is thus to detect entanglement in a two-particle reduced state $\varrho_{ab} = \mathrm{tr}_{(a,b)^c}(\varrho)$ for only one pair (a, b) with the complement $(a, b)^c$. This can be achieved by accessing only the two-body correlations as minimal information.

The notion of spin squeezing originally relies on certain spin-squeezing parameters [61], but in several previous works [44,45,62–64], a state ϱ is called *spin-squeezed* if its entanglement can be detected from the values of $\langle J_l \rangle$ and $\langle J_l^2 \rangle$ only for any three orthogonal directions, e.g., $l = x,\ y,\ z$. For a symmetric state, its spin-squeezing entanglement has been completely characterized in a *necessary and sufficient* manner by proving the entanglement in the two-particle reduced states [44,45,62–64]. On the other hand, such a characterization requires optimizations over collective measurement directions for a given quantum state.

In the following, we will show that the collective randomized measurement scheme can reach the same conclusion without such an optimization. We can formulate the first main result of this Letter:

Observation 1.—For an N-qubit permutationally symmetric state ϱ, the first, second, and third moments $\mathcal{J}^{(r)}(\varrho)$ for $r = 1,\ 2,\ 3$ completely characterize spin-squeezing entanglement. That is, a constructive procedure for achieving the necessary and sufficient condition is obtained by the moments with the parameters $\alpha = 2/N_2$, $\beta = -2(N-2)/(NN_2)$, $\gamma = -1/[2(N-1)]$ and $N_2 = N(N-1)$.

The proof of this observation is given in Appendix A in the Supplemental Material [65]. As the proof's main idea, we will first explain the known fact that a necessary and sufficient condition is equivalent to the violation of $C \geq 0$ for the covariance matrix $C_{ij} = \langle \sigma_i \otimes \sigma_j \rangle_{\varrho_{ab}} - \langle \sigma_i \rangle_{\varrho_a} \langle \sigma_j \rangle_{\varrho_b}$, with the reduced state ϱ_{ab} for any choice a, b, for details; see Refs. [54,81,82]. Then we will analytically show that the violation can be determined from the moments $\mathcal{J}^{(r)}(\varrho)$ for $r = 1,\ 2,\ 3$, and we will provide an explicit procedure to decide spin-squeezing entanglement.

We remark that any N-qubit permutationally symmetric state can be given by a density matrix in the so-called Dicke basis. For m excitations, the Dicke state is defined as $|D_{N,m}\rangle = \binom{N}{m}^{-1/2} \sum_k \pi_k (|1\rangle^{\otimes m} \otimes |0\rangle^{\otimes (N-m)})$, where the summation is over the different permutations of the qubits and m is an integer such that $0 \leq m \leq N$. A concrete example is the state $|D_{3,1}\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. Then states mixed from Dicke, W, and GHZ (Greenberger–Horne–Zeilinger) states are permutationally symmetric. Accordingly, Observation 1 allows us to detect such spin-squeezed GME states in a CRFI manner.

Multiparticle bound entanglement—Next, let us consider the more general case where ϱ is not permutationally symmetric. Even in this case, our approach with collective randomized measurements is effective for detecting spin-squeezing entanglement. We can present the second result in this Letter:

Observation 2.—For an N-qubit state ϱ, the first moment $\mathcal{J}^{(1)}$ with $(\alpha, \beta, \gamma) = (3, 0, 0)$ is given by

$$\mathcal{J}^{(1)}(\varrho) = \sum_{l=x,y,z} (\Delta J_l)^2. \qquad (7)$$

Any N-qubit fully separable state obeys

$$\mathcal{J}^{(1)}(\varrho) \geq \frac{N}{2}. \qquad (8)$$

Then violation implies the presence of multipartite entanglement.

The proof of this observation is given in Appendix B of the Supplemental Material. The criterion in Eq. (8) itself was already established [83], so we only have to show the derivation of Eq. (7). While the proof employs Weingarten calculus [84] to evaluate the Haar integrals, we note that the relation in Eq. (7) can be understood in the context of quantum designs. These allow to replace unitary integrals of certain polynomials by finite sums. In the present case, one would require spherical two designs [2,5,85].

The criterion in Eq. (8) can be maximally violated by the so-called many-body spin singlet states $\varrho_{\mathrm{singlet}}$ [44,45,83,86–90]. A pure singlet state is defined as a state invariant under any collective unitary: $U^{\otimes N}|\Psi_{\mathrm{singlet}}\rangle = e^{i\theta}|\Psi_{\mathrm{singlet}}\rangle$. That is, it is simultaneous eigenstates of J_l for $l = x,\ y,\ z$ with zero eigenvalue. Many-body spin singlet states $\varrho_{\mathrm{singlet}}$ are mixtures of pure singlet states and are also invariant under any collective local unitary: $U^{\otimes N} \varrho_{\mathrm{singlet}} (U^\dagger)^{\otimes N} = \varrho_{\mathrm{singlet}}$. Since the state $\varrho_{\mathrm{singlet}}$ has $\langle J_l^k \rangle = 0$ for and any integer k, it is at the center of the collective Bloch sphere (see Fig. 1).

Moreover, the criterion of Eq. (8) is known to be a very strong entanglement condition. In fact, it can detect the so-called multiparticle bound entanglement [44,45], which can be positive under partial transposition (PPT) for all bipartitions [91,92]. We stress that Observation 2 only requires second moments over Haar unitary integrals. This shows that collective randomized measurements are fundamentally different from previous randomized measurements [6,7].

In addition, we mention that Observation 2 can be used in probing many-body Bell nonlocality [93,94] and improving quantum metrology [95–97]. Also, we will discuss its high-dimensional generalizations in Appendix B of the Supplemental Material. Finally, we remark that the criterion in Eq. (8) is known as one of the optimal inequalities to detect spin-squeezing entanglement [44,45].

Antisymmetric entanglement—So far, we have considered the moments $\mathcal{J}^{(r)}(\varrho)$ based on the function $f_U(\varrho)$ in Eq. (3). Since $f_U(\varrho)$ contained two-body quantum correlations via $\langle J_z^2 \rangle_U$ and is related to the collective angular momenta as symmetric observables, we detected entanglement with large two-body correlations and certain symmetries, such as Dicke and singlet states. In the following, we will develop collective randomized measurements to analyze quantum systems in terms of nonsymmetric observables with three-body correlations.

To proceed, let us begin by considering the three-qubit observable $\mathcal{S}(\sigma_x \otimes \sigma_y \otimes \sigma_z) \equiv \sigma_x \otimes \sigma_y \otimes \sigma_z + \sigma_y \otimes \sigma_z \otimes \sigma_x + \sigma_z \otimes \sigma_x \otimes \sigma_y + \cdots$, where $\mathcal{S}$ denotes the average over all permutations of indices x, y, z. This observable is invariant under any particle exchange: $\mathbb{S}_{ab}\mathcal{S}(\sigma_x \otimes \sigma_y \otimes \sigma_z)\mathbb{S}_{ab} = \mathcal{S}(\sigma_x \otimes \sigma_y \otimes \sigma_z)$, with $\mathbb{S}_{ab}$ being the SWAP operator for any a, b. The N-qubit extension of this observable can be represented by the product of collective angular momenta

$$\mathcal{O}_{\mathcal{S}} \equiv \sum_{i<j<k} \mathcal{S}\left(\sigma_x^{(i)} \otimes \sigma_y^{(j)} \otimes \sigma_z^{(k)}\right) = \frac{8}{3!}\mathcal{S}(J_x J_y J_z), \quad (9)$$

where $\mathcal{S}(J_x J_y J_z) = J_x J_y J_z + J_y J_z J_x + J_z J_x J_y + \cdots$ and $\mathbb{S}_{ab}\mathcal{O}_{\mathcal{S}}\mathbb{S}_{ab} = \mathcal{O}_{\mathcal{S}}$. In general, any combination of products of collective angular momenta remains permutationally invariant under particle exchange [98].

An associated operator with $\mathcal{O}_{\mathcal{S}}$ from Eq. (9) is the antisymmetric observable

$$\mathcal{O}_{\mathcal{A}} \equiv \sum_{i<j<k} \mathcal{A}\left(\sigma_x^{(i)} \otimes \sigma_y^{(j)} \otimes \sigma_z^{(k)}\right), \quad (10)$$

where $\mathcal{A}(\sigma_x \otimes \sigma_y \otimes \sigma_z)$ denotes the antisymmetrization of $\sigma_x \otimes \sigma_y \otimes \sigma_z$ by taking the sum over even permutations and subtracting the sum over odd permutations of indices x, y, z. Clearly, $\mathcal{O}_{\mathcal{A}}$ cannot be constructed from collective angular momenta.

We have seen that symmetric observables based on collective angular momenta can detect symmetric entanglement with collective randomized measurements. Then, one may wonder if antisymmetric observables such as $\mathcal{O}_{\mathcal{A}}$ can characterize antisymmetric entanglement. Similarly to Eq. (4), we define the average over random collective local unitaries as follows:

$$\mathcal{T}(\varrho) = \int dU \mathrm{tr}[\varrho U^{\otimes N} \mathcal{O}_{\mathcal{A}} (U^\dagger)^{\otimes N}]. \quad (11)$$

For this we can formulate the third result in this Letter:

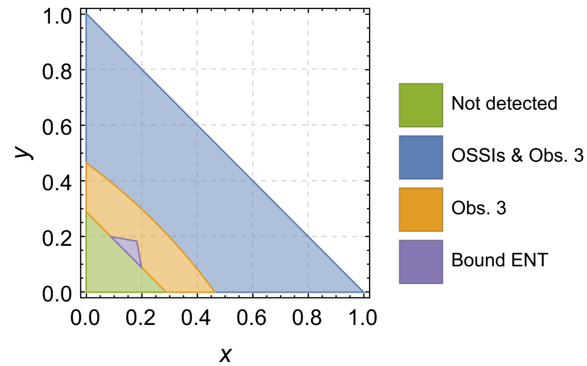

FIG. 2. Entanglement criteria for the mixed state in Eq. (14) for $N = 3$ in the x-y plane. The fully separable states are contained in green area, which obeys all the optimal spin-squeezing inequalities (OSSIs) previously known with optimal measurement directions [44,45] and also our criterion in Observation 3. The blue area corresponds to the spin-squeezed entangled states that can be detected by all OSSIs and Observation. 3. The yellow and purple areas correspond to the entangled states that cannot be detected by all OSSIs but can be detected by Observation 3, thus marking the improvement of this Letter compared with previous results. In particular, the purple area corresponds to the multiparticle bound entangled states that are not detected by the PPT criterion for all bipartitions but detected by Observation 3.

Observation 3.—The average $\mathcal{T}(\varrho)$ is given by

$$\mathcal{T}(\varrho) = \mathrm{tr}[\varrho \mathcal{O}_{\mathcal{A}}] = \sum_{i<j<k} \sum_{a,b,c} \varepsilon_{abc} \left\langle \sigma_a^{(i)} \otimes \sigma_b^{(j)} \otimes \sigma_c^{(k)} \right\rangle_{\varrho}, \quad (12)$$

where ε_{abc} denotes the Levi-Civita symbol for $a, b, c = x$, y, z. Any N-qubit fully separable state can obey a certain tight bound,

$$|\mathcal{T}(\varrho)| \leq p_{\mathrm{fs}}^{(N)}, \quad (13)$$

where $p_{\mathrm{fs}}^{(N)}$ can be computed analytically for $N = 3$ and numerically for up to $N \leq 7$ and is, up to numerical precision, given by $p_{\mathrm{fs}}^{(N)} = N^2 \cot(\pi/N)/3\sqrt{3}$. Then violation implies the presence of multipartite entanglement.

The derivation of Eq. (12) and the explanation of Eq. (13) are given in Appendix C of the Supplemental Material. Also, we will analytically show that any three-qubit biseparable state obeys $|\mathcal{T}(\varrho)| \leq 2$; see Appendix C, where this violation signals GME states.

Let us test our criterion with the two-parameter family of states

$$\varrho_{x,y} = x|\zeta_N\rangle\langle\zeta_N| + y|\tilde{\zeta}_N\rangle\langle\tilde{\zeta}_N| + \frac{1-x-y}{2^N}\mathbb{1}_2^{\otimes N}, \quad (14)$$

where $0 \leq x, y \leq 1$. Here, $|\zeta_N\rangle$ is the so-called phased Dicke state [99–105], up to normalization,

$$|\zeta_N\rangle = \sum_{k=1}^{N} \frac{e^{\frac{2\pi i k}{N}}}{\sqrt{N}} |0\rangle_1 |0\rangle_2 \cdots |1\rangle_k \cdots |0\rangle_{N-1} |0\rangle_N, \quad (15)$$

and the state $|\tilde{\zeta}_N\rangle = \sigma_x^{\otimes N}|\zeta_N\rangle$ with $\langle\zeta_N|\tilde{\zeta}_N\rangle = 0$. Note that the phased Dicke state is not equivalent to the Dicke state $|D_{N,1}\rangle$ under collective unitary transformations. In Fig. 2, we illustrate the criterion of Observation 3 for the state $\varrho_{x,y}$ for $N = 3$ on the x-y plane. Note that the subspace spanned by $|\zeta_3\rangle$ and $|\tilde{\zeta}_3\rangle$ is, after noncollective local unitaries, equivalent to the maximally entangled subspace of three qubits, as characterized in Ref. [106].

Our result allows us to detect entangled states that cannot be detected not only for Eq. (8) but also for all the other optimal spin-squeezing inequalities previously known with optimal measurement directions [44,45]. Moreover, the multipartite bound entanglement of $\varrho_{x,y}$ can be also detected. For $N \geq 4$, similar results are obtained, see Appendix C in the Supplemental Material.

The inequality (13) can be maximally violated by several GME states. For small N, we have numerically confirmed that $|\zeta_N\rangle$ and $|\tilde{\zeta}_N\rangle$ can reach the maximal violation, that is, they can be the eigenstates with the largest singular values of $\mathcal{O}_A$. For $N = 3$, the eigenvalue decomposition of $\mathcal{O}_A$ is given by $\mathcal{O}_A = 2\sqrt{3}(|\zeta_3\rangle\langle\zeta_3| + |\tilde{\zeta}_3\rangle\langle\tilde{\zeta}_3| - |\mu_3\rangle\langle\mu_3| - |\tilde{\mu}_3\rangle\langle\tilde{\mu}_3|)$, where $|\mu_3\rangle$ is a state obtained by changing $e^{(2\pi ik/3)}$ in $|\zeta_3\rangle$ to $e^{[2\pi i(4-k)/3]}$ and $|\tilde{\mu}_3\rangle = \sigma_x^{\otimes 3}|\mu_3\rangle$. Here, all the eigenstates are mutually orthogonal, and the dimension of this eigensubspace is four, which coincides with the maximal dimension of a three-qubit completely entangled subspace that contains no full product state [107,108]. Finally, we mention that for cases with $N = 4, 5, 6$, the matrix rank of $\mathcal{O}_A$ is respectively given by 6,24,38.

Entanglement between two ensembles—Let us apply the strategy of collective randomized measurements to another scenario where two ensembles are spatially separated [109–114]. We denote ϱ_{AB} as a $2N$-qubit state that contains the two ensembles of N spin-$\frac{1}{2}$ particles, where $\varrho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_X = \mathcal{H}_2^{\otimes N}$ for $X = A, B$. Supposing that each ensemble can be controlled individually, we can perform the collective randomized measurements to obtain the moments $\mathcal{J}_X^{(r)}$ with a fixed choice (α, β, γ). Note that a related approach to detect entanglement between two spin ensembles has been discussed in Ref. [115].

The total collective observables are given by $J_l^{\pm} = J_{l,A} \pm J_{l,B}$, where $J_{l,X} = \frac{1}{2}\sum_{i=1}^N \sigma_l^{(X_i)} \in \mathcal{H}_X$ for $l = x, y, z$ and Pauli matrices $\sigma_l^{(X_i)}$ acting on X_ith subsystem in the ensemble $X = A, B$. Note that one can also formulate the following result considering a more general case: $J_{k,l}^{\pm} = J_{k,A} \pm J_{l,B}$. In a similar manner to Eq. (2b), we can introduce the random variances $(\Delta J_z^{\pm})^2_{U_{AB}}$ with $U_{AB} = U_A \otimes U_B$. Denoting the gap as $\eta_{U_{AB}} \equiv (\Delta J_z^{+})^2_{U_{AB}} - (\Delta J_z^{-})^2_{U_{AB}}$, let us consider its moment

$$\mathcal{G}_{AB}^{(r)} = g \int dU_{AB} [\eta_{U_{AB}}]^r, \tag{16}$$

where g is a real constant parameter.

Now we can present the following criterion:

Observation 4.—For a $2N$-qubit state ϱ_{AB} with the permutationally symmetric reduced states, any separable ϱ_{AB} obeys

$$\mathcal{G}_{AB}^{(2)} + \mathcal{J}_A^{(1)} + \mathcal{J}_B^{(1)} - \mathcal{J}_A^{(1)}\mathcal{J}_B^{(1)} \leq 1, \tag{17}$$

where $g = (3/N^2)^2$ and $(\alpha, \beta, \gamma) = (0, 12/N^2, 0)$.

The proof is given in Appendix D of the Supplemental Material. As the proof's main idea, we will first simply evaluate the integrals on the left-hand side in Eq. (17). Then we will adopt the separability criterion presented in Ref. [16] (see, Proposition 5 there) in order to find the entanglement criterion in Eq. (17).

The violation of this inequality allows us to detect entanglement between the spatially separated two ensembles. In Appendix D, we will demonstrate how the criterion in Eq. (17) can characterize entanglement between two ensembles. Also, we will show that Observation 4 can be extended to the case of m ensembles for $m \geq 3$.

Statistically significant tests—Finally, we note that the statistical analysis of collective randomized measurements is discussed in Appendix E in the Supplemental Material. There, we will provide estimations for the necessary number of measurements required for entanglement detection with high confidence. Similar discussions can also be found in Refs. [2,8–10,15].

Conclusion—We have introduced the concept of collective randomized measurements as a tool for CRFI quantum information processing. Based on the framework, we have proposed systematic methods to characterize quantum correlations. In particular, we showed that our approach has detected spin-squeezing entanglement, multipartite bound entanglement, and spatially separated entanglement in two ensembles.

There are several directions for future research. First, it would be interesting to extend our work to higher-order scenarios involving J_l^k for $k > 2$. Such extensions may facilitate various connections, such as nonlinear spin squeezing [116] or permutationally invariant Bell inequalities [117–120]. Next, the inequality (13) resembles multipartite entanglement witnesses [121]. Exploring this may lead to more advanced techniques for analyzing multipartite entanglement. Finally, while the standard version of randomized measurements has found many applications beyond entanglement detection, e.g., in quantum metrology, shadow tomography, or cross-platform verification [1,2], one may study these possibilities also for the collective randomized measurements introduced here.

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