

**Entanglement measures  
(How much is it entangled?)  
(Lecture of the Quantum Information class of  
the Master in Quantum Science and  
Technology)**

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## 1 Entanglement measures (How much is it entangled?)

- Motivation
- A. General quantum operation
- B. Local operations and classical communication (LOCC)
- C. Entanglement of formation
- D. Concurrence
- E. Entanglement of distillation
- F. Bound entanglement
- G. Requirements for entanglement measures
- H. Negativity

# Entanglement measures

- After detecting entanglement, we have to ask how entangled the state is.
- It will turn out that entanglement is a resource.

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# General quantum operation

- The general quantum operation is defined as

$$\rho' = \sum_k E_k \rho E_k^\dagger$$

with

$$\sum_k E_k^\dagger E_k = 1.$$

- $E_k$  are Kraus operators.
- Generalized measurements, POVM (positive operator-valued measure).
- Special case: von Neumann measurements, when  $E_k$  are pairwise orthogonal projectors.
- Naimark's dilation theorem:  
general operation =  
von Neumann measurement on system+ancilla.

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# Local operations and classical communication (LOCC)

- LOCC are
  - local unitaries,
  - local von Neumann or POVM measurements,
  - local unitaries or measurements conditioned on measurement outcomes on the other party.
- Mathematical description of LOCC. Separable operations are a somewhat larger set, however, this set can easily be described.

$$\rho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \rho \left( E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger$$

with

$$\sum_k \left( E_k^{(1)} \otimes E_k^{(2)} \right)^\dagger \left( E_k^{(1)} \otimes E_k^{(2)} \right) = 1.$$

# Local operations and classical communication (LOCC) II

- Stochastic Local Operations and Classical Communication (SLOCC):

$$|\Psi\rangle' \leftarrow E_k^{(1)} \otimes E_k^{(2)} |\Psi\rangle$$

It happens with some probability, not deterministic.

- LOCC cannot create entanglement. Separable states remain separable under LOCC.
- LOCC can create correlations.



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# Entropy of entanglement

- The von Neumann entropy is defined as

$$S(\rho) = -\text{Tr}(\rho \log_2 \rho).$$

- It can be written with the eigenvalues of the density matrix as

$$S(\rho) = -\sum_{k=1}^d \lambda_k \log_2 \lambda_k.$$

- For a pure state we have  $\lambda_k = \{1, 0, 0, \dots, 0\}$ , and thus it is zero.
- Its maximal is for the completely mixed state for which  $\lambda_k = \{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\}$ , and its value is  $\log_2 d$ .
- For a bipartite pure state, the **entropy of entanglement** is

$$E_E(|\Psi\rangle) = S(\text{Tr}_1(|\Psi\rangle\langle\Psi|)).$$

That is, it is the von Neumann entropy of the reduced state is an entanglement measure.

# Entropy of entanglement II

- Comments
  - It is one for two-qubit singlet states.
  - It is zero for product states.
  - It is invariant under  $U_1 \otimes U_2$ .

# Entanglement of formation

- For mixed states, the entanglement of formation is the convex roof of the von Neumann entropy of the reduced state.

$$E_F = \min_{\{|\Psi_k\rangle, p_k\}} \sum_k p_k E_E(|\Psi_k\rangle),$$

- The optimization is over all decompositions of the state of the type

$$\varrho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- $E_F$  tells us, in the asymptotic limit, how many singlets we need to create the state.
- Is it easy to compute? No. For  $2 \times 2$  systems, there is an explicit formula with the concurrence. For larger systems, there is not a general method.

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# Entanglement of formation

- For two qubits,  $E_F$  can be calculated explicitly (Wootters, 1997).
- Special case: for pure states the concurrence is

$$C(|\Psi\rangle) = |\langle\Psi|\tilde{\Psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|,$$

where

$$|\Psi\rangle = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}.$$

- It is related to the linear entropy of the reduced state.

$$C = \sqrt{2(1 - \text{Tr}(\rho_{\text{red}}^2))}, \quad (1)$$

where

$$\rho_{\text{red}} = \text{Tr}_2(|\Psi\rangle\langle\Psi|). \quad (2)$$

# Entanglement of formation II

- Now we have to compute  $E_F$  from  $C$ .
- We also need that

$$\epsilon(c) = H_2\left(\frac{1 + \sqrt{1 - c^2}}{2}\right).$$

Here

$$H_2 = -x \log_2 x - (1 - x) \log_2(1 - x).$$

- Then,  $E_F$  can be obtained as

$$E_F(\rho) = \epsilon(C(\rho)).$$

## Entanglement of formation III

- For mixed states, the **concurrence** is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where  $\lambda_k$ 's are, in a decreasing order, the eigenvalues of

$$R = \sqrt{\sqrt{\rho} \tilde{\rho} \sqrt{\rho}},$$

and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y).$$



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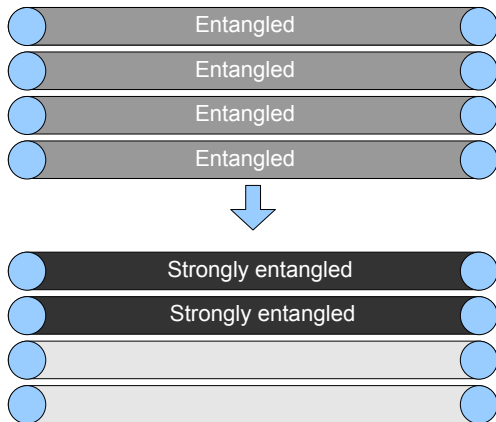
# Entanglement of distillation

- $E_D$  tells us, how many singlets we can obtain from the state with LOCC. In general,

$$E_F \geq E_D.$$

- Note that local operation and classical communication means that we have several copies and we can act on the copies locally.

# Entanglement of distillation II



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# Bound entanglement

- There are states that need entangled particles to be created, but singlets cannot be distilled from them.
- All PPT entangled states are like that. (That is, all entangled states that are not detected by the Peres-Horodecki criterion.)

## Bound entanglement II

- Next, we will prove this. First we show that PPT state remain PPT under LOCC. Under LOCC we have

$$\varrho' = \sum_k E_k^{(1)} \otimes E_k^{(2)} \varrho (E_k^{(1)} \otimes E_k^{(2)})^\dagger$$

We also have

$$(\varrho')^{T2} = \sum_k E_k^{(1)} \otimes ((E_k^{(2)})^\dagger)^T \varrho^{T2} (E_k^{(1)})^\dagger \otimes (E_k^{(2)})^T$$

Here we used that  $(AB)^T = B^T A^T$  and  $A^\dagger = (A^*)^T$ .

- We can see that if  $\varrho^{T2} \geq 0$  then  $(\varrho')^{T2} \geq 0$ . Thus the PPT states remain PPT under LOCC.

R., P., M., and K. Horodecki, *Rev. Mod. Phys.* 81, 865 (2009).

(Click on the link above, see "G. Bound entanglement - when distillability fails" on page 44.)

# Bound entanglement III

- Let us again remember the flip operator

$$F|k\rangle|l\rangle = |l\rangle|k\rangle$$

It has eigenvalues  $\pm 1$ .

- The maximally entangled state

$$|\Psi_{\text{me}}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d |k\rangle|k\rangle.$$

## Bound entanglement IV

- We can show that

$$|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}| = \frac{1}{d} \sum_{k,l} |k\rangle\langle l| \otimes |k\rangle\langle l|,$$

$$|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|^{T_1} = \frac{1}{d} \sum_{k,l} |k\rangle\langle l| \otimes |l\rangle\langle k| \equiv \frac{F}{d},$$

where  $|l\rangle$  are real orthonormal vectors.

- Now we show that PPT states have a small overlap with the maximally entangled state. For PPT states, the fidelity with respect to the maximally entangled state is

$$\text{Tr}(|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|\varrho) = \text{Tr}(|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}|^{T_1}\varrho^{T_1}) = \frac{1}{d} \text{Tr}(F\varrho^{T_1}) \leq \frac{1}{d},$$

since  $\varrho^{T_1} \geq 0$  and  $F$  has  $\pm 1$  eigenvalues.



## Bound entanglement IV

- Thus, PPT states have a small fidelity with respect to the maximally entangled state. Even LOCC operations cannot increase this.
- A simple product state can reach  $1/d$

$$\text{Tr}(|\Psi_{\text{me}}\rangle\langle\Psi_{\text{me}}||11\rangle\langle 11|) = \frac{1}{d}.$$

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# Requirements for entanglement measures

- 1 To each density matrix it assigns a nonnegative number. Typically, the maximally entangled state has  $\log d$ .
- 2  $E(\rho) = 0$  for separable states.
- 3  $E$  does not increase on average under LOCC.

$$E(\rho) \leq \sum_k p_k E\left(\frac{A_k \rho A_k^\dagger}{\text{Tr}(A_k \rho A_k^\dagger)}\right), \quad A_k = E_k^{(1)} \otimes E_k^{(2)}.$$

- 4 For pure states, it has the same value as the entanglement entropy.
- Entanglement monotone: 1,2,3.
  - Entanglement measure: 1,2,4 and does not increase under deterministic LOCC, i.e.,

$$E(\rho') \leq E(\rho); \quad \rho' = \sum_k A_k \rho A_k^\dagger, \quad A_k = E_k^{(1)} \otimes E_k^{(2)}, \quad (\text{POVM}).$$

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# Trace norm

- Let us consider the singular decomposition of a matrix

$$A = U\Sigma V^\dagger,$$

where

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_d)$$

and  $\sigma_k > 0$ .

- Then the trace norm is

$$\|A\|_1 = \text{Tr}\left(\sqrt{AA^\dagger}\right) = \sum_k \sigma_k.$$

- The Hilbert-Schmidt norm is

$$\|A\|_2 = \text{Tr}\left(AA^\dagger\right) = \sum_k \sigma_k^2.$$

# Negativity

- Example for a monotone: negativity

$$N(\varrho) = \frac{\|\varrho^{T1}\| - 1}{2}.$$

Trace norm= sum of singular values.

- For Hermitian matrices, it is the same as sum of eigenvalues.

$$N(\varrho) = \frac{\sum_k |\lambda_k| - 1}{2}.$$

- Note that  $\sum_k \lambda_k = 1$ . Then, assume that the first  $M$  eigenvalues are negative, the rest is positive. We get

$$N(\varrho) = \frac{\sum_{k=1}^M -\lambda_k + \sum_{k=M+1}^d \lambda_k - \sum_k \lambda_k}{2}.$$

## Negativity II

- Hence,

$$N(\varrho) = \sum_{k=1}^M |\lambda_k|.$$

That is, the absolute value of the sum of the negative eigenvalues of the partial transpose.

- Clearly, it is zero for PPT states. Thus, it is zero for all separable states.
- Not as meaningful as the Entanglement of Formation, but can be calculated on any system sizes.
- It fulfills certain conditions on how it changes under LOCC. It does not increase under deterministic LOCC.