# Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles

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# Outline

# Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea.
- In many cases, we need to detect bipartite entanglement.

# Outline

# Entanglement

**A state is (fully) separable if it can be written as**

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes ... \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

# Outline

# Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \frac{1}{2}\sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

# Outline

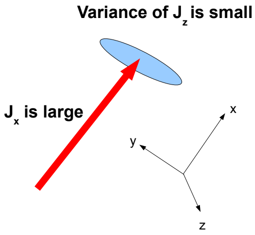
# The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_s^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_s^2 < 1$ then the state is entangled.
- States detected are like this:




- They are good for metrology!

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1. Motivation
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$
3. Detecting multipartite entanglement of Dicke states
   - Dicke state realized with a BEC of two-state atoms
4. Detecting bipartite entanglement of Dicke states
   - Bipartite entanglement from multipartite entanglement in BEC
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   - Entanglement detection in Dicke states
   - Our criteria work under realistic conditions

# Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

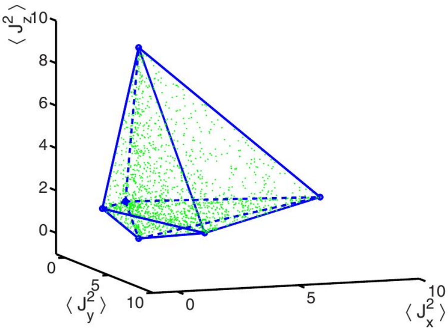
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \tfrac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \tfrac{N}{2}, \qquad \text{(singlet states)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \tfrac{N}{2}, \qquad \text{(Dicke states)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \tfrac{N(N-2)}{4},$$

where $k, l, m$ take all the possible permutations of $x, y, z$.

singlets: GT, Phys. Rev. A 69, 052327 (2004);
all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-$j$: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

# Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope




- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

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# Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and $J_z$.
- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left( |0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

- Summing over all permutations.
- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left( |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.,* PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

# Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

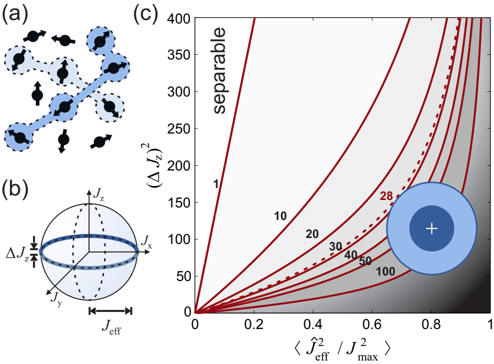
Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.

- ... are macroscopically entangled, like GHZ states.

Fröwis, Dür, PRL 2011.

# Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.




$$J^2_{\mathrm{eff}} = J^2_x + J^2_y, \quad J_{\max} = \frac{N}{2}.$$


B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014).

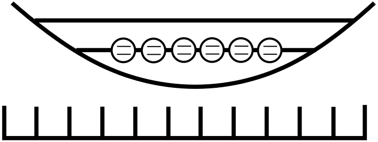
# Outline

# Bipartite entanglement from bosonic multipartite entanglement

- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

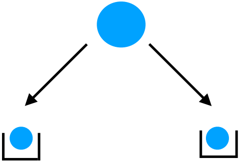
# Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument



See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)

$|n_0 = 1\rangle |n_1 = 1\rangle$

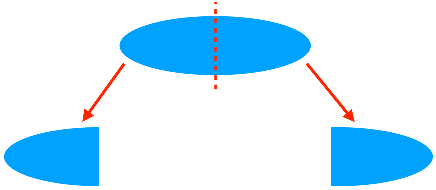


$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

# Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.

- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

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# Experiment in the group of Carsten Klempt at the University of Hannover

- Rubidium BEC, spin-1 atoms.

- Initially all atoms in the spin state $|j_z = 0\rangle$.

- Dynamics
$$H = a_0^2 a_{+1}^\dagger a_{-1}^\dagger + (a_0^\dagger)^2 a_{+1} a_{-1}.$$
Tunneling from mode 0 to the mode $+1$ and $-1$.

- Two-particle example:
$$|j_z = 0\rangle|j_z = 0\rangle \to \frac{1}{\sqrt{2}}(|j_z = +1\rangle|j_z = -1\rangle + |j_z = -1\rangle|j_z = +1\rangle)$$
$$= \text{Dicke state of 2 particles.}$$

# Experiment in the group of Carsten Klempt at the University of Hannover II

- After some time, we have a state

$$|n_0, n_{-1}, n_{+1}\rangle = |N - 2n, n, n\rangle.$$

- That is, $N - 2n$ particles remained in the $|j_z = 0\rangle$ state, while $2n$ particles form a symmetric Dicke state given as

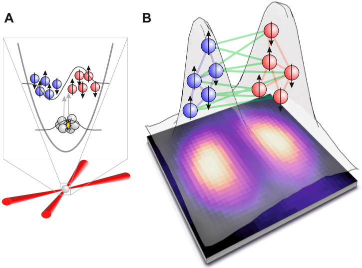
$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}}\right),$$

  where we use $|0\rangle$ and $|1\rangle$ instead of $|j_z = -1\rangle$ and $|j_z = +1\rangle$.

- Half of the atoms in state $|0\rangle$, half of the atoms in state $|1\rangle$ + symmerization.

# Experiment in the group of Carsten Klempt at the University of Hannover III

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.




K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, G. Tóth, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

# Correlations for Dicke states

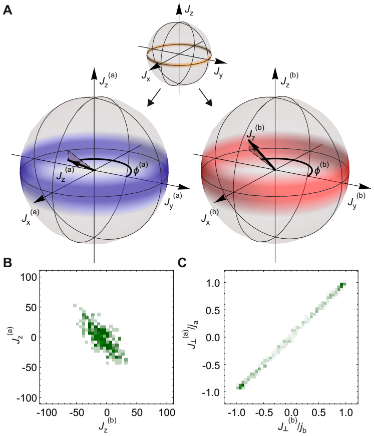
- For the Dicke state

$$\begin{aligned}(\Delta(J_x^a - J_x^b))^2 &\approx 0,\\ (\Delta(J_y^a - J_y^b))^2 &\approx 0,\\ (\Delta J_z)^2 = (\Delta(J_z^a + J_z^b))^2 &= 0.\end{aligned}$$

- Measurement results on well "*b*" can be predicted from measurements on "*a*"

$$\begin{aligned} J_x^b &\approx J_x^a, && \text{(correlation)}\\ J_y^b &\approx J_y^a, && \text{(correlation)}\\ J_z^b &= -J_z^a. && \text{(anti-correlation)}\end{aligned}$$

# Correlations for Dicke states - experimental results




Here, $J_\perp^{(n)} = \cos\alpha J_x^{(n)} + \sin\alpha J_y^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

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# Number-phase-like uncertainty

- We start from the sum of two Heisenberg uncertainty relations

$$(\Delta J_z)^2[(\Delta J_x)^2+(\Delta J_y)^2] \geq \frac{1}{4}(\langle J_x\rangle^2+\left\langle J_y\right\rangle^2).$$

Then,

$$(\Delta J_z)^2[(\Delta J_x)^2+(\Delta J_y)^2]+\frac{1}{4}[(\Delta J_x)^2+(\Delta J_y)^2] \geq \frac{1}{4}(\left\langle J_x^2\right\rangle+\left\langle J_y^2\right\rangle).$$

- Simple algebra yields

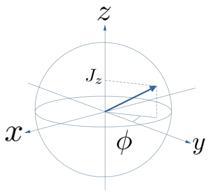
$$\left[(\Delta J_z)^2+\frac{1}{4}\right]\times\frac{(\Delta J_x)^2+(\Delta J_y)^2}{\langle J_x^2\rangle+\langle J_y^2\rangle} \geq \frac{1}{4}.$$

- Note that $\langle J_x^2\rangle$ appears, not $\langle J_x\rangle^2$.

# Number-phase-like uncertainty II

- Uncertainty relation

$$\underbrace{\left[(\Delta J_z)^2 + \frac{1}{4}\right]}_{\sim \text{ fluctuation of } J_z} \times \underbrace{\frac{(\Delta J_x)^2 + (\Delta J_y)^2}{\langle J_x^2 \rangle + \langle J_y^2 \rangle}}_{\sim \text{ phase fluctuation}} \geq \frac{1}{4}.$$




Handwaving description:
$J_z$ and $\phi$ cannot be defined both with high accuracy.

# Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_x^n = \frac{J_x^n}{\sqrt{j_n(j_n+1)}}, \quad \mathcal{J}_y^n = \frac{J_y^n}{\sqrt{j_n(j_n+1)}},$$

  where $n = a, b$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables → resistance to experimental imperfections.

# Uncertainty with normalized variables

Our uncertainty relation is now

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x)^2 + (\Delta \mathcal{J}_y)^2\right] \geq \frac{1}{4}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle .$$

# The two-well EPR-Steering criterion

## Main result I

For states with a hidden state model,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{4}\left\langle(\mathcal{J}_x^a)^2 + (\mathcal{J}_y^a)^2\right\rangle^2$$

holds. $|D_N\rangle : \frac{1}{4}$ $\approx \frac{4}{N}$ $\approx \frac{1}{4}$ $\frac{LHS}{RHS} = \frac{4}{N}$

Any state violating the inequality cannot be described by a hidden state model, i.e., the state is *steerable*.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b. \end{aligned}$$

# The two-well entanglement criterion

## Main result II

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{J}_x^-)^2 + (\Delta \mathcal{J}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{J}_x^2 + \mathcal{J}_y^2 \right\rangle^2$$

holds. $|D_N\rangle : \frac{1}{4}$ $\approx \frac{4}{N}$ $\approx 1$ $\frac{LHS}{RHS} = \frac{1}{N}$

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{J}_x^- &= \mathcal{J}_x^a - \mathcal{J}_x^b, \\ \mathcal{J}_y^- &= \mathcal{J}_y^a - \mathcal{J}_y^b. \end{aligned}$$

# Outline

# Problem 1: Varying particle number

- The experiment is repeated many times. Each time we find a somewhat different particle number.

- Postselecting for a given particle number is not feasible.

- Density matrix:

$$\varrho = \sum_{N_a,N_b} Q_{N_a,N_b} \varrho_{N_a,N_b},$$

  where $Q_{N_a,N_b}$ are probabilities and $\varrho_{N_a,N_b}$ are states.

- $\varrho$ is entangled iff at least one of the $\varrho_{N_a,N_b}$ is entangled.

# Problem 1: Varying particle number II

- Splitting noise: splitting is not pefect.
- $\langle N_a \rangle = \langle N_b \rangle = \frac{N}{2}$.
- $N_a = \frac{N}{2} + x$, $N_b = \frac{N}{2} - x$, $x$=inbalance.
- Particle number variance

$$(\Delta N_a)^2 = (\Delta x)^2 = \langle x^2 \rangle = \frac{N}{4}, \qquad (\Delta N_a) = \frac{\sqrt{N}}{2}$$

- Variance of collective observable

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}, \quad l = x, y.$$

Twice as large due to the unequal splitting.

# Problem 1: Varying particle number III

- $N/2 : N/2$ splitting:

$$[\Delta(J_l^a - J_l^b)]^2 = \frac{N}{8}, \quad l = x, y.$$

- Real splitting with partition noise:

$$[\Delta(J_l^a - J_l^b)]^2 \approx \frac{N}{4}.$$

# Problem 1: Varying particle number IV

- Solution: we normalize the measured quantities

$$\mathcal{J}_l^- = \frac{1}{\sqrt{j_a(j_a+1)}} J_l^a - \frac{1}{\sqrt{j_b(j_b+1)}} J_l^b$$

for $l = x, y$.

- We obtain

$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{N}{N^2/2 + 4N - 2x^2}.$$

- After splitting $|x| \lesssim \sqrt{N/4}$.

- Hence,

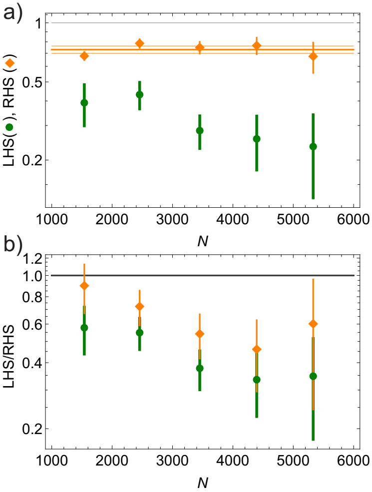
$$(\Delta \mathcal{J}_l^-)^2 \approx \frac{2}{N}.$$

$(\Delta \mathcal{J}_l)^2$ is not sensitive to the fluctuation of $x$ if $N$ is large.

# Problem 2: States are not always symmetric in a BEC of two-state atoms

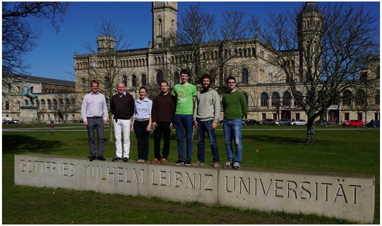
- Ideally, the BEC is in a single spatial mode.
- Hence, the state of an ensemble of the two-state atoms must be symmetric.
- In practice, the BEC is not in a single spatial mode, hence the state is no perfectly symmetric.
- Our criterion must hadle this.

# Violation of the criterion: entanglement is detected II




LHS/RHS for (top) our present work, and (bottom) for Science 2018.

# Collaborators on entanglement conditions for double-well Dicke states

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# Summary

- Detection of bipartite entanglement and EPR steering close to Dicke states. It works also for split spin-squeezed states.
- Non-symmetric states within the wells and a varying particle number can also be handled.

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann, B. Lücke, C. Klempt, and G. Tóth,
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