Detecting metrologically useful multipartite entanglement in cold gases

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Strait of Gibraltar
Ceuta (SPAIN)
Alborán Sea
Melilla (SPAIN)
Oran
Algiers
Tunis
Crete
Rabat
Casablanca
MOROCCO
ALGERIA
TUNISIA
Valletta
MALTA
Scale 1: 19,500,000
Lambert Conformal Conic Projection, standard parallels 40°N and 56°N
0 300 Kilometers
0 20 300 Miles

Outline

1. **Motivation**
   - Why entanglement is important?
2. **Spin squeezing and entanglement**
   - Entanglement
   - Collective measurements
   - The original spin-squeezing criterion
   - Generalized criteria for $j = \frac{1}{2}$, singlet states
3. **Detecting multipartite entanglement of Dicke states**
   - Dicke state realized with a BEC of two-state atoms
4. **Detecting bipartite entanglement of Dicke states**
   - Bipartite entanglement from multipartite entanglement in BEC
   - Dicke state in a double well
   - Entanglement detection in Dicke states
5. **Quantum metrology**
   - Quantum Fisher information and entanglement
   - Experimental tests

Why multipartite entanglement is important?

- Many experiments are aiming to create entangled states with many atoms.
- Full tomography is not possible, we still have to say something meaningful.
- Only collective quantities can be measured.
- Thus, entanglement detection seems to be a good idea ...

Why is this challenging?

- It could happen that
  - it is not possible to create large scale entanglement in a system that is not completely isolated.
  - such entanglement is created, but we cannot verify its presence, since we can measure few things.

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Entanglement

A state is (fully) separable if it can be written as

$$\sum_k p_k \varrho_1^{(k)} \otimes \varrho_2^{(k)} \otimes \ldots \otimes \varrho_N^{(k)}.$$

If a state is not separable then it is entangled.

k-producibility/k-entanglement

A pure state is k-producible if it can be written as

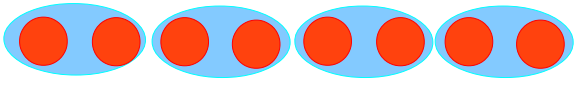
$$|\Phi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \otimes |\Phi_3\rangle \otimes |\Phi_4\rangle ....$$

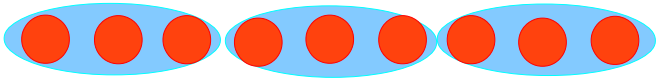
where $|\Phi_l\rangle$ are states of at most k qubits.

A mixed state is k-producible, if it is a mixture of k-producible pure states.

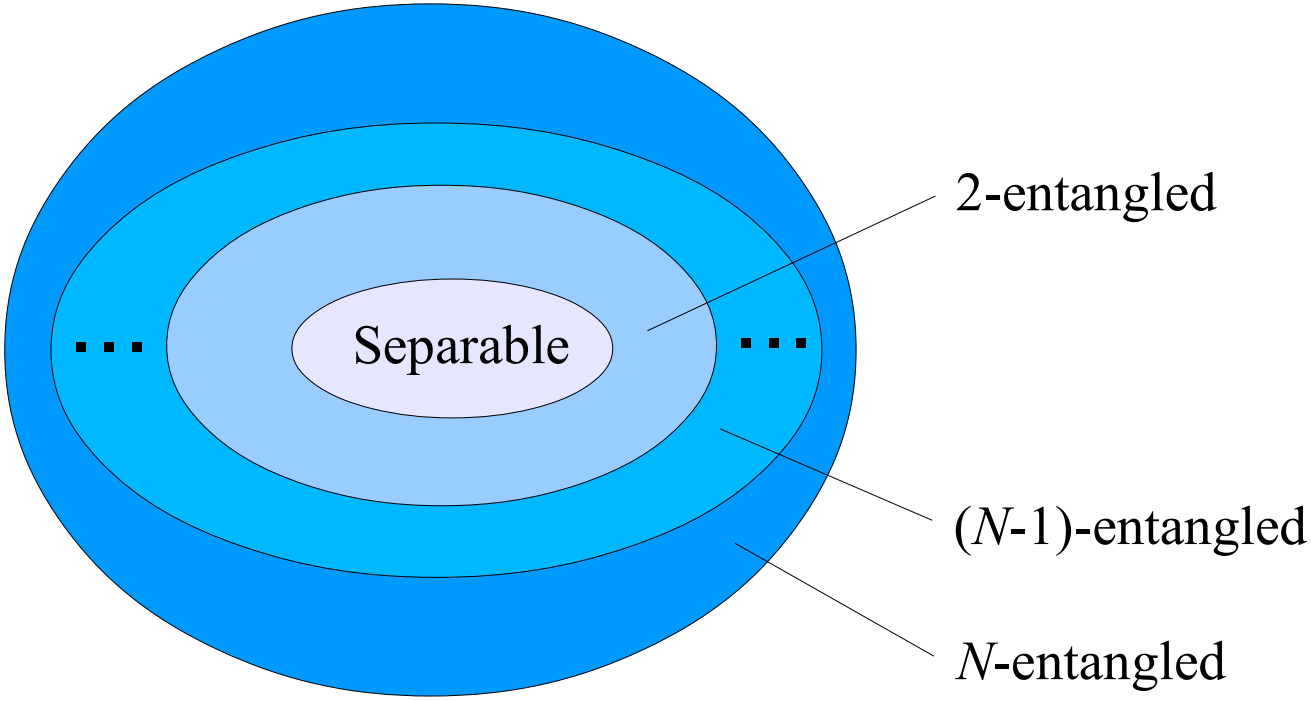
e.g., O. Gühne and GT, New J. Phys 2005.

- If a state is not k-producible, then it is at least $(k+1)$-particle entangled.





k-producibility/k-entanglement II



$(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$ 2-entangled

$(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$ 3-entangled

$(|0000\rangle + |1111\rangle) \otimes (|0\rangle + |1\rangle)$ 4-entangled

k-entanglement means real k-particle quantumness

- k-entanglement means that we could not make trivially the experiment from $(k-1)$-particle experiments.
- The state is not a mixture of product states

$$\varrho_1 \otimes \varrho_2 \otimes \varrho_3 \otimes ...$$

such that all ϱ_l has at most $(k-1)$ qubits.

Other possibility

- High dimensional bipartite entanglement

  M. Krenn, M. Huber, R. Fickler, R. Lapkiewicz, S. Ramelow, A. Zeilinger, PNAS 111, 6243 (2014).

  S Liu, M Fadel, Q He, M Huber, G Vitagliano, Quantum 8, 1236 (2024).

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Many-particle systems for j=1/2

- For spin-$\frac{1}{2}$ particles, we can measure the collective angular momentum operators:

$$J_l := \tfrac{1}{2} \sum_{k=1}^{N} \sigma_l^{(k)},$$

where $l = x, y, z$ and $\sigma_l^{(k)}$ a Pauli spin matrices.

- We measure the expectation values $\langle J_l \rangle$.

- We can also measure the variances

$$(\Delta J_l)^2 := \langle J_l^2 \rangle - \langle J_l \rangle^2.$$

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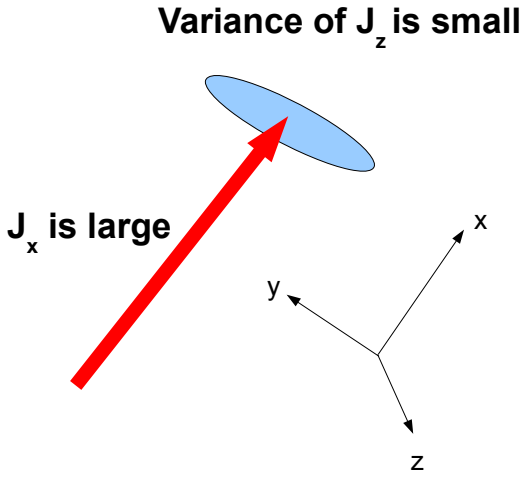
The standard spin-squeezing criterion

The spin squeezing criterion for entanglement detection is

$$\xi_S^2 = N \frac{(\Delta J_z)^2}{\langle J_x \rangle^2 + \langle J_y \rangle^2}.$$

[A. Sørensen, L.M. Duan, J.I. Cirac, P. Zoller, Nature 409, 63 (2001).]

- If $\xi_S^2 < 1$ then the state is entangled.
- States detected are like this:



- They are good for metrology!

Multipartite entanglement in spin squeezing

- We consider pure k-producible states of the form

$$|\Psi\rangle = \otimes_{l=1}^{M} |\psi_l\rangle,$$

where $|\psi_l\rangle$ is the state of at most k qubits.

Extreme spin squeezing

The spin-squeezing criterion for k-producible states is

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right),$$

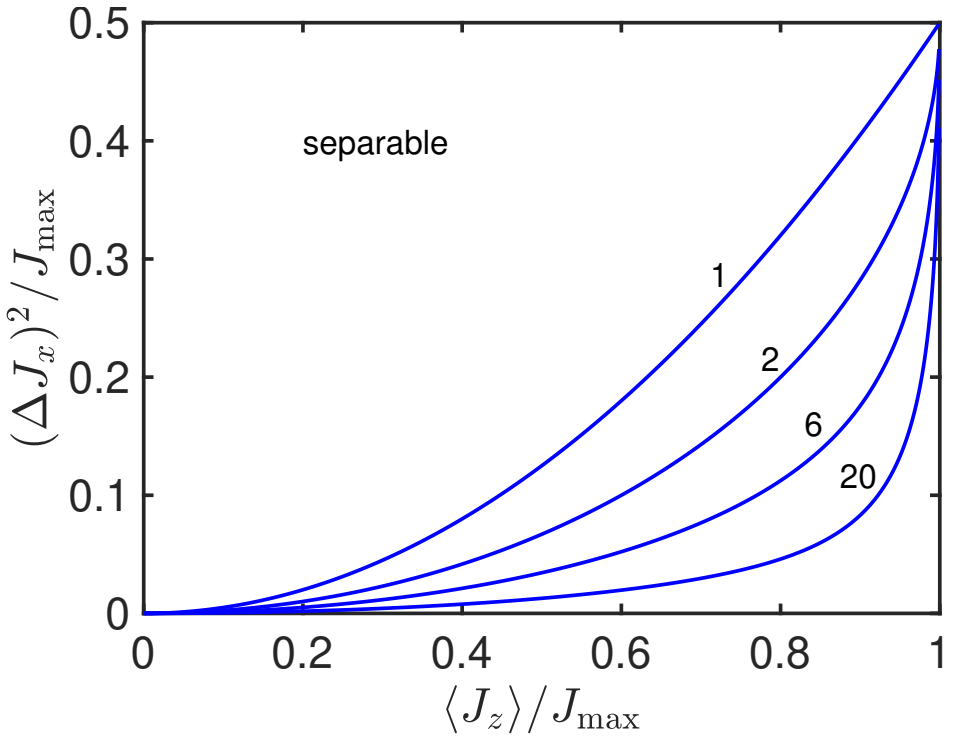
where $J_{\max} = \frac{N}{2}$ and we use the definition

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle}{j} = X} (\Delta j_z)^2.$$

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

Multipartite entanglement in spin squeezing II

- Larger and larger multipartite entanglement is needed to larger and larger squeezing ("extreme spin squeezing").



- $N = 100$ spin-1/2 particles, $J_{\max} = N/2$.

Sørensen and Mølmer, Phys. Rev. Lett. 86, 4431 (2001); experimental test: Gross, Zibold, Nicklas, Esteve, Oberthaler, Nature 464, 1165 (2010).

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Generalized spin squeezing criteria for $j = \frac{1}{2}$

- Let us assume that for a system we know only

$$\vec{J} := (\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle),$$
$$\vec{K} := (\langle J_x^2 \rangle, \langle J_y^2 \rangle, \langle J_z^2 \rangle).$$

- Then any state violating the following inequalities is entangled:

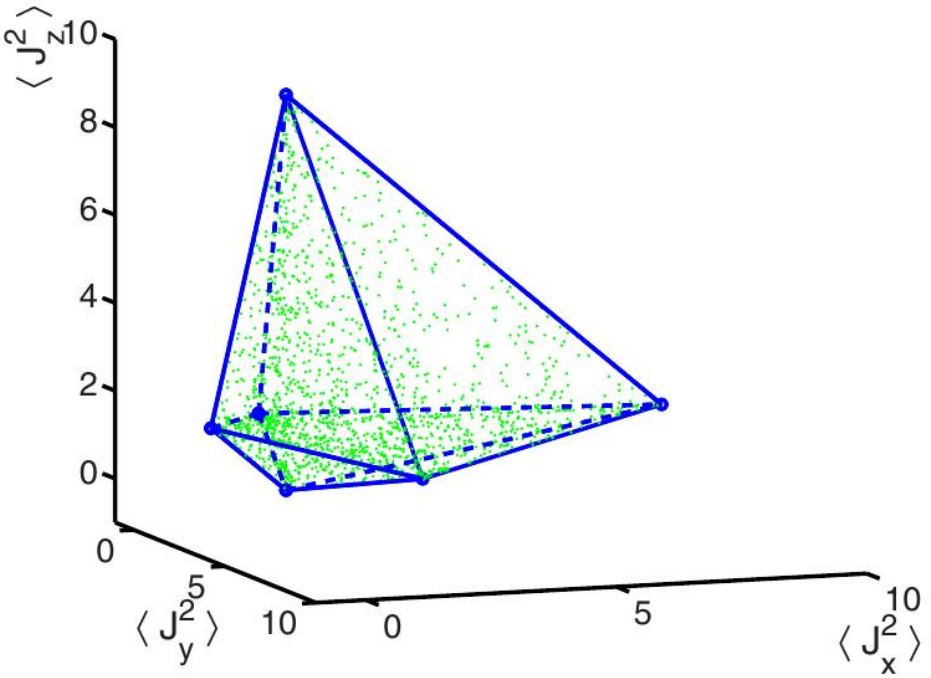
$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4},$$
$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}, \qquad \text{(singlet states)}$$
$$\langle J_k^2 \rangle + \langle J_l^2 \rangle \leq (N-1)(\Delta J_m)^2 + \frac{N}{2}, \qquad \text{(Dicke states)}$$
$$(N-1)\left[(\Delta J_k)^2 + (\Delta J_l)^2\right] \geq \langle J_m^2 \rangle + \frac{N(N-2)}{4},$$

where k, l, m take all the possible permutations of x, y, z.

singlets: GT, Phys. Rev. A 69, 052327 (2004);
all Eqs.: GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);
spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Generalized spin squeezing criteria for $j = \frac{1}{2}$ II

- Separable states are in the polytope



- We set $\langle J_l \rangle = 0$ for $l = x, y, z$.

Spin squeezing criteria – Two-particle correlations

All quantities depend only on two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Average 2-particle density matrix

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- We can detect states with a separable $\varrho_{2\mathrm{p}}$!

Singlet state

- Singlet states are ground states of antiferromagnetic Hamiltonians.

- The permutationally invariant singlet is

$$\varrho_{\text{singlet}} \propto \lim_{T\to 0} e^{-\frac{J_x^2+J_y^2+J_z^2}{T}}.$$

- For such a state, for large N we have

$$\varrho_{2\text{p}} \approx \frac{\mathbb{1}}{4},$$

still it is detected as entangled by our criterion!

- Such a state has been created in cold atoms.

Singlet state experiments

- Creating singlets by squeezing the uncertainties of spin components → incoherent process, "creating entanglement with decoherence".

G. Tóth and M. W. Mitchell, Generation of macroscopic singlet states in atomic ensembles, New J. Phys. 12 053007 (2010).

- Entanglement in singlet states in $> 10^6$ cold Rb atoms.

N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, GT, R. J. Sewell, and M. W. Mitchell, Generation of Macroscopic Singlet States in a Cold Atomic Ensemble, Phys. Rev. Lett. 113, 093601 (2014).

- Entanglement in singlet states in $> 10^{13}$ hot alkali atoms.

J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, GT, and Morgan W. Mitchell, Measurement-induced, spatially-extended entanglement in a hot, strongly-interacting atomic system,
Nat. Commun. 11, 2415 (2020).

Applications of singlets

- Singlets are invariant under homogeneous fields.
- Gradient field destroys the singlet → Gradient metrology.

I. Apellaniz, I. Urizar-Lanz, Z. Zimborás, P. Hyllus, and GT, Precision bounds for gradient magnetometry with atomic ensembles, Phys. Rev. A 97, 053603 (2018).

S. Altenburg, M. Oszmaniec, S. Wölk, O. Gühne, Estimation of gradients in quantum metrology, Phys. Rev. A 96, 042319 (2017).

I. Urizar-Lanz, P. Hyllus, I. L. Egusquiza, M. W. Mitchell, and GT, Macroscopic singlet states for gradient magnetometry, Phys. Rev. A 88, 013626 (2013).

Entanglement conditions based on the two-body density matrix

- Spin squeezing conditions with collective variables based on the two-body density matrix:

Spin Squeezing Inequalities and Entanglement of *N* Qubit States, J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. 95, 120502 (2005).

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where k, l, m take all the possible permutations of x, y, z.

GT, C. Knapp, O. Gühne, and H.J. Briegel, PRL 99, 250405 (2007);

spin-j: G. Vitagliano, P. Hyllus, I. L. Egusquiza, GT, PRL 107, 240502 (2011).

Bose-Einstein condensate people

Netflix movie "Spectral"

Filmed in Budapest

Dicke states

- Dicke states: eigenstates of $\vec{J}^2 = J_x^2 + J_y^2 + J_z^2$ and J_z.

- Symmetric Dicke states of spin-1/2 particles, with $\langle J_z \rangle = \langle J_z^2 \rangle = 0$

$$|D_N\rangle = \binom{N}{\frac{N}{2}}^{-\frac{1}{2}} \sum_k \mathcal{P}_k \left(|0\rangle^{\otimes \frac{N}{2}} \otimes |1\rangle^{\otimes \frac{N}{2}} \right).$$

  Summing over all permutations.
  Due to symmetry, $\left\langle \vec{J}^2 \right\rangle$ is maximal.

- E.g., for four qubits they look like

$$|D_4\rangle = \frac{1}{\sqrt{6}} \left(|0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle \right).$$

photons: N. Kiesel, C. Schmid, GT, E. Solano, H. Weinfurter, PRL 2007; Prevedel. *et al.*, PRL 2009; W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, GT, H. Weinfurter, PRL 2009.

cold atoms: Lücke, Science 2011; Hamley *et al*, Nat. Phys. 2012.

Dicke states are useful because they ...

- ... possess strong multipartite entanglement, like GHZ states.

GT, JOSAB 2007.

- ... are optimal for quantum metrology, similarly to GHZ states.

Hyllus *et al.*, PRA 2012; Lücke *et al.*, Science 2011;
GT, PRA 2012;
GT and Apellaniz, J. Phys. A, special issue for “50 year of Bell’s theorem”, 2014.

- ... are macroscopically entangled, like GHZ states.

Fröwis, Dür, PRL 2011.

Spin Squeezing Inequality for Dicke states

- Let us rewrite the third inequality. For separable states

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_z)^2$$

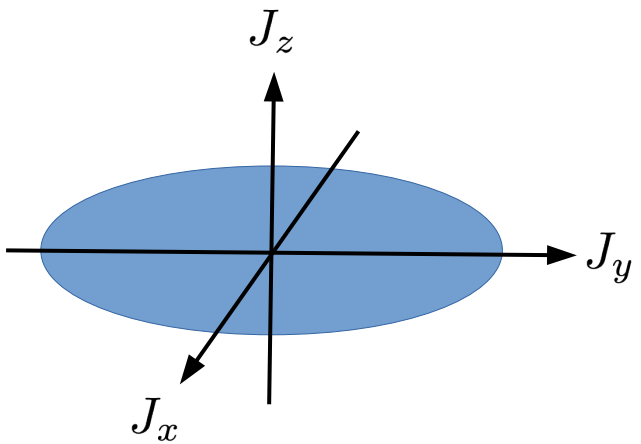
holds.

- It detects states close to Dicke states since

$$\langle J_x^2 + J_y^2 \rangle = \frac{N}{2}\left(\frac{N}{2}+1\right) = \max.,$$

$$\langle J_z^2 \rangle = 0.$$

- "Pancake" like uncertainty ellipse.



Multipartite entanglement around Dicke states

- Measure the same quantities as before

$$(\Delta J_z)^2$$

and

$$\langle J_x^2 + J_y^2 \rangle.$$

- In contrast, for the original spin-squeezing criterion we measured $(\Delta J_z)^2$ and $\langle J_x \rangle^2 + \langle J_y \rangle^2$.

- Pioneering work: linear condition of Luming Duan, Phys. Rev. Lett. (2011). See also Zhang, Duan, New. J. Phys. (2014).

Multipartite entanglement - Our condition

- Sørensen-Mølmer condition for k-producible states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x \rangle^2 + \langle J_y \rangle^2}}{J_{\max}} \right).$$

- Combine it with

$$\langle J_x^2 + J_y^2 \rangle \leqslant J_{\max}(\tfrac{k}{2} + 1) + \langle J_x \rangle^2 + \langle J_y \rangle^2,$$

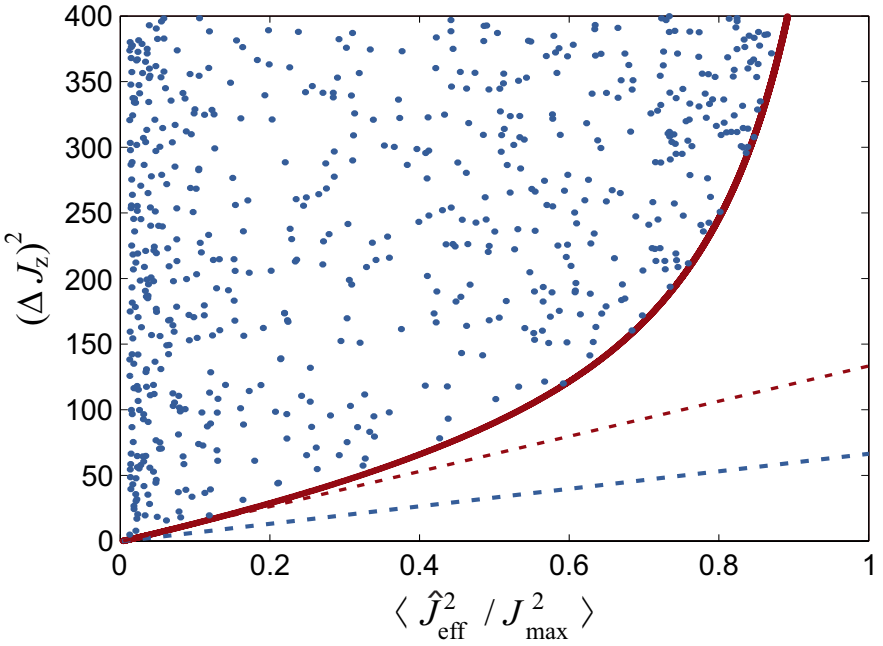
which is true for pure k-producible states. (Remember, $J_{\max} = \frac{N}{2}$.)

Condition for entanglement detection around Dicke states

$$(\Delta J_z)^2 \geqslant J_{\max} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - J_{\max}(\frac{k}{2} + 1)}}{J_{\max}} \right).$$

Due to convexity properties of the expression, this is also valid to mixed separable states.

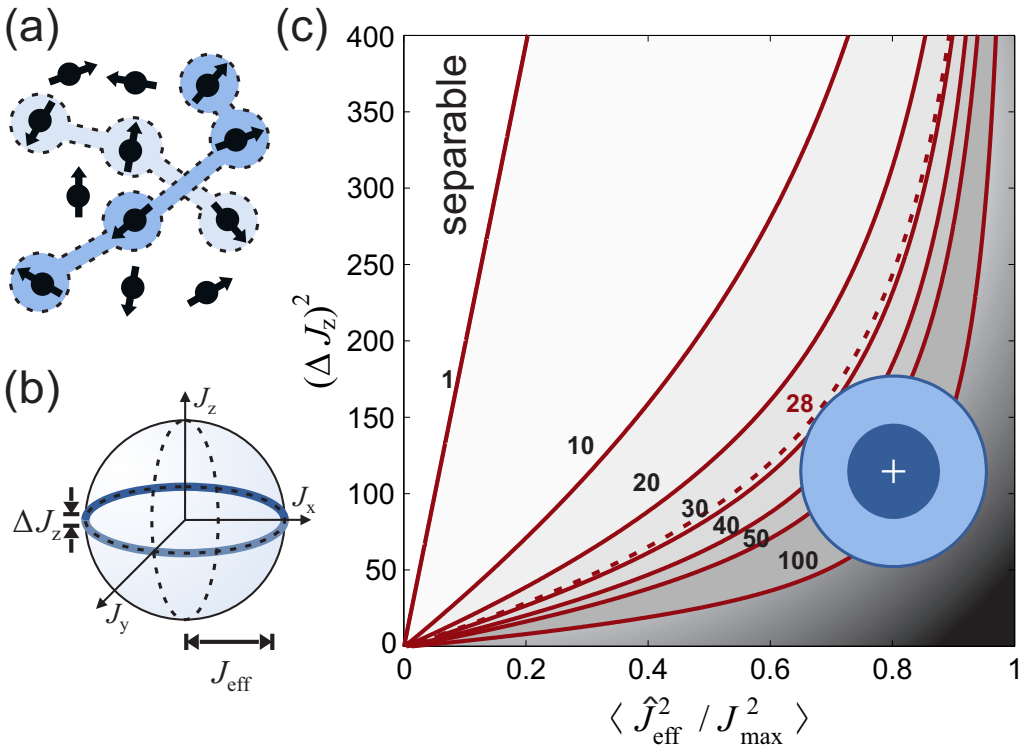
Concrete example



- $N = 8000$ particles, and $J_{\rm eff} = J_x^2 + J_y^2$.
- Red curve: boundary for 28-particle entanglement.
- Blue dashed line: linear condition given in
  L.-M. Duan, Phys. Rev. Lett. 107, 180502 (2011).
- Red dashed line: tangent of our curve.

Multipartite entanglement

- Bose-Einstein condensate, 8000 particles. 28-particle entanglement is detected.



$$J^2_{\rm eff} = J^2_x + J^2_y, \quad J_{\rm max} = \frac{N}{2}.$$

B. Lücke, J. Peise, G. Vitagliano, J. Arlt, L. Santos, GT, and C. Klempt, PRL 112, 155304 (2014); G. Vitagliano *et al.*, New J. Phys. 19, 013027 (2017).

Multipartite entanglement

- Multi-particle entanglement:

  Yi-Quan Zou *et al.,* PNAS 115 (25) 6381-6385 (2018),

  Lin Xin *et al.,* Phys. Rev. Lett. 131, 133402 (2023).

- Two-mode squeezing:

  N. Bar-Gill, C. Gross, I. Mazets, M. Oberthaler, and G. Kurizki, Phys. Rev. Lett. 106, 120404 (2011).

Spin squeezing criteria – Two-particle correlations

All quantities depend only on two-particle correlations

$$\langle J_l \rangle = N \langle j_l \otimes \mathbb{1} \rangle_{\varrho_{2\mathrm{p}}}; \quad \langle J_l^2 \rangle = \frac{N}{4} + N(N-1) \langle j_l \otimes j_l \rangle_{\varrho_{2\mathrm{p}}}.$$

- Average 2-particle density matrix

$$\varrho_{2\mathrm{p}} = \frac{1}{N(N-1)} \sum_{n \neq m} \varrho_{mn}.$$

- We can even detect multipartite entanglement knowing only two-body correlations!

Outline

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   - Generalized criteria for $j = \frac{1}{2}$, singlet states
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   - Dicke state realized with a BEC of two-state atoms
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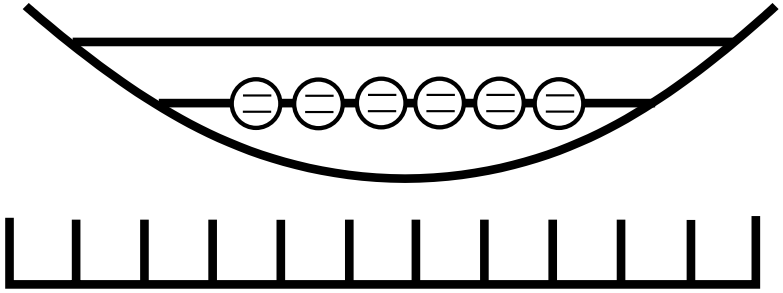
Bipartite entanglement from bosonic multipartite entanglement

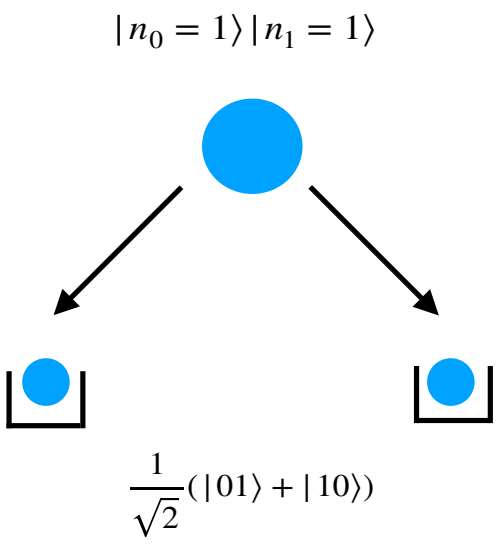
- In the BEC, "all the particles are at the same place."
- In the usual formulation, entanglement is between spatially separated parties.
- Is multipartite entanglement within a BEC useful/real?
- Answer: yes!

Bipartite entanglement from bosonic multipartite entanglement II

- Dilute cloud argument

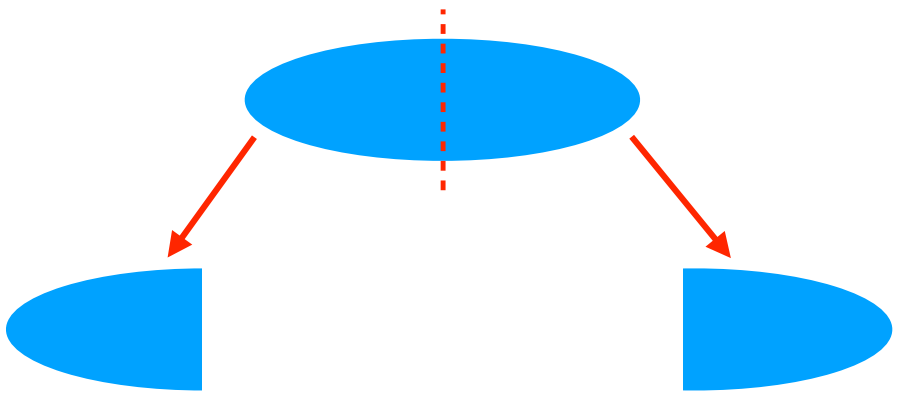


See, e.g., P. Hyllus, L. Pezzé, A Smerzi and GT, PRA 86, 012337 (2012)



Bipartite entanglement from bosonic multipartite entanglement III

- After splitting it into two, we have bipartite entanglement if we had before multipartite entanglement.
- The splitting does not generate entanglement, if we consider projecting to a fixed particle number.



N. Killoran, M. Cramer, and M. B. Plenio, PRL 112, 150501 (2014).

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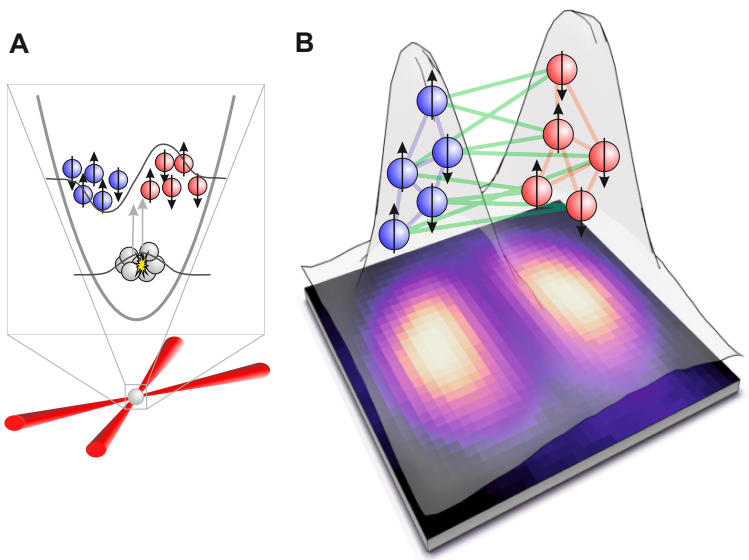
Experiment in the group of Carsten Klempt at the University of Hannover

- Important: first excited spatial mode of the trap was used, not the ground state mode.
- It has two "bumps" rather than one, hence they had a split Dicke state.



K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, and C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

Correlations for Dicke states

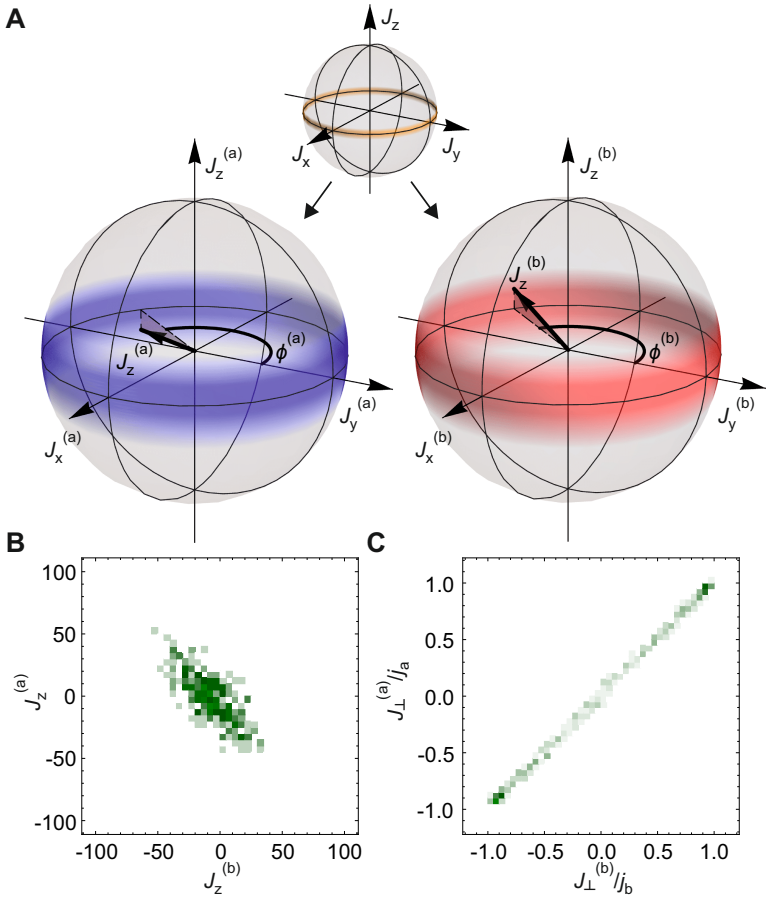
- For the Dicke state

$$
\begin{aligned}
(\Delta(J_{\mathrm{x}}^{a} - J_{\mathrm{x}}^{b}))^2 &\approx 0, \\
(\Delta(J_{\mathrm{y}}^{a} - J_{\mathrm{y}}^{b}))^2 &\approx 0, \\
(\Delta J_{\mathrm{z}})^2 &= 0.
\end{aligned}
$$

- Measurement results on well "b" can be predicted from measurements on "a"

$$
\begin{aligned}
J_{\mathrm{x}}^{b} &\approx J_{\mathrm{x}}^{a}, \\
J_{\mathrm{y}}^{b} &\approx J_{\mathrm{y}}^{a}, \\
J_{\mathrm{z}}^{b} &= -J_{\mathrm{z}}^{a}.
\end{aligned}
$$

Correlations for Dicke states - experimental results



Here, $J_{\perp}^{(n)} = \cos\alpha J_{x}^{(n)} + \sin\alpha J_{y}^{(n)}$.

Experiment in K. Lange *et al.*, Science 334, 773–776 (2011).

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Normalized variables

- Let us introduce the normalized variables

$$\mathcal{J}_m^n = \frac{J_m^n}{\sqrt{j_n(j_n+1)}} \approx \frac{J_m^n}{N_n},$$

where $m = x, y$ and $n = a, b$ (i.e., left well, right well), the total spin is

$$j_n = \frac{N_n}{2},$$

- Normalized variables $\rightarrow$ resistance to experimental imperfections.

The two-well entanglement criterion

Suggestion of the experimentalists: we need a product criterion, since it is good for realistic noise.

Main result I

For separable states,

$$\left[(\Delta J_z)^2 + \frac{1}{4}\right]\left[(\Delta \mathcal{I}_x^-)^2 + (\Delta \mathcal{I}_y^-)^2\right] \geq \frac{1}{16}\left\langle \mathcal{I}_x^2 + \mathcal{I}_y^2 \right\rangle^2$$

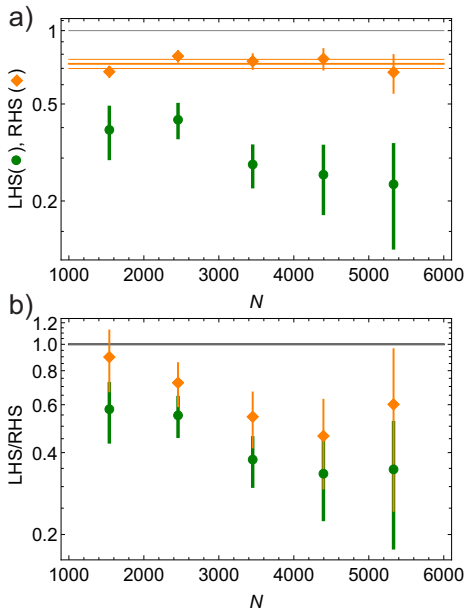
holds.

Here,

$$\begin{aligned} J_z &= J_z^a + J_z^b, \\ \mathcal{I}_m^- &= \mathcal{I}_m^a - \mathcal{I}_m^b \end{aligned}$$

for $m = x, y$.

Violation of the criterion: entanglement is detected II



LHS/RHS (top) for Quantum 2023, and (bottom) for Science 2018.

K. Lange, J. Peise, B. Lücke, I. Kruse, G. Vitagliano, I. Apellaniz, M. Kleinmann, GT, C. Klempt, Entanglement between two spatially separated atomic modes, Science 360, 416 (2018).

G. Vitagliano, I. Apellaniz, M. Fadel, M. Kleinmann, B. Lücke, C. Klempt, GT, Number-phase uncertainty relations and bipartite entanglement detection in spin ensembles, Quantum 7, 914 (2023).

Recent related works published back to back

- Spatially distributed multipartite entanglement enables EPR steering of atomic clouds:

  P. Kunkel, M. Prüfer, H. Strobel, D. Linnemann, A. Frölian, T. Gasenzer, M. Gärttner, M. K. Oberthaler,
  Science 360, 413-416 (2018).

- Spatial entanglement patterns and Einstein-Podolsky-Rosen steering in Bose-Einstein condensates:

  M. Fadel, T. Zibold, B. Décamps, P. Treutlein,
  Science 360, 409-413 (2018).

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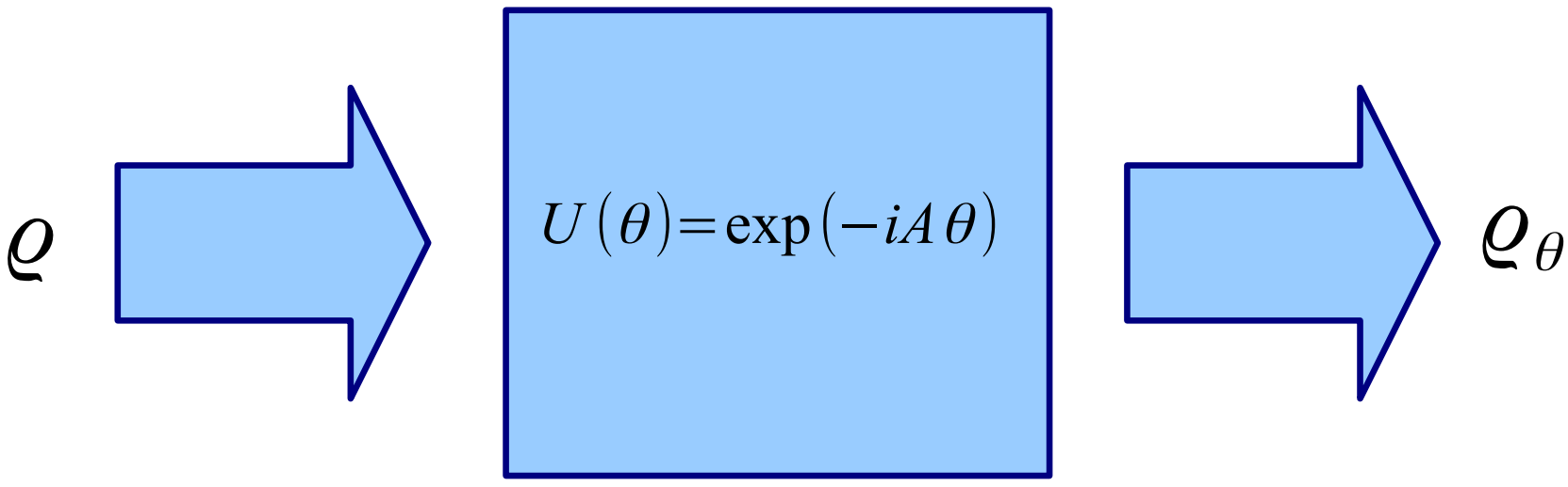
Quantum metrology

- Fundamental task in metrology



- We have to estimate θ in the dynamics

$$U = \exp(-iA\theta).$$

The quantum Fisher information

- Cramér-Rao bound on the precision of parameter estimation

$$(\Delta\theta)^2 \geq \frac{1}{\nu F_Q[\varrho, A]},$$

where $F_Q[\varrho, A]$ is the quantum Fisher information, and ν is the number of independent repetitions.

- The quantum Fisher information is

$$F_Q[\varrho, A] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|A|l\rangle|^2,$$

where $\varrho = \sum_k \lambda_k |k\rangle\langle k|$.

Linear interferometers

- We consider

$$F_Q[\varrho, J_l]$$

  for $l = x, y, z$.

- The Hamiltonian is J_l, which does not contain interactions.

The quantum Fisher information vs. entanglement

- For separable states of N spin-1/2 particles (qubits)

$$F_Q[\varrho, J_l] \leq N, \qquad l = x, y, z.$$

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009);

Hyllus, Gühne, Smerzi, Phys. Rev. A 82, 012337 (2010).

- For states with at most k-qubit entanglement (tight bound if k is a divisor of N)

$$F_Q[\varrho, J_l] \leq kN.$$

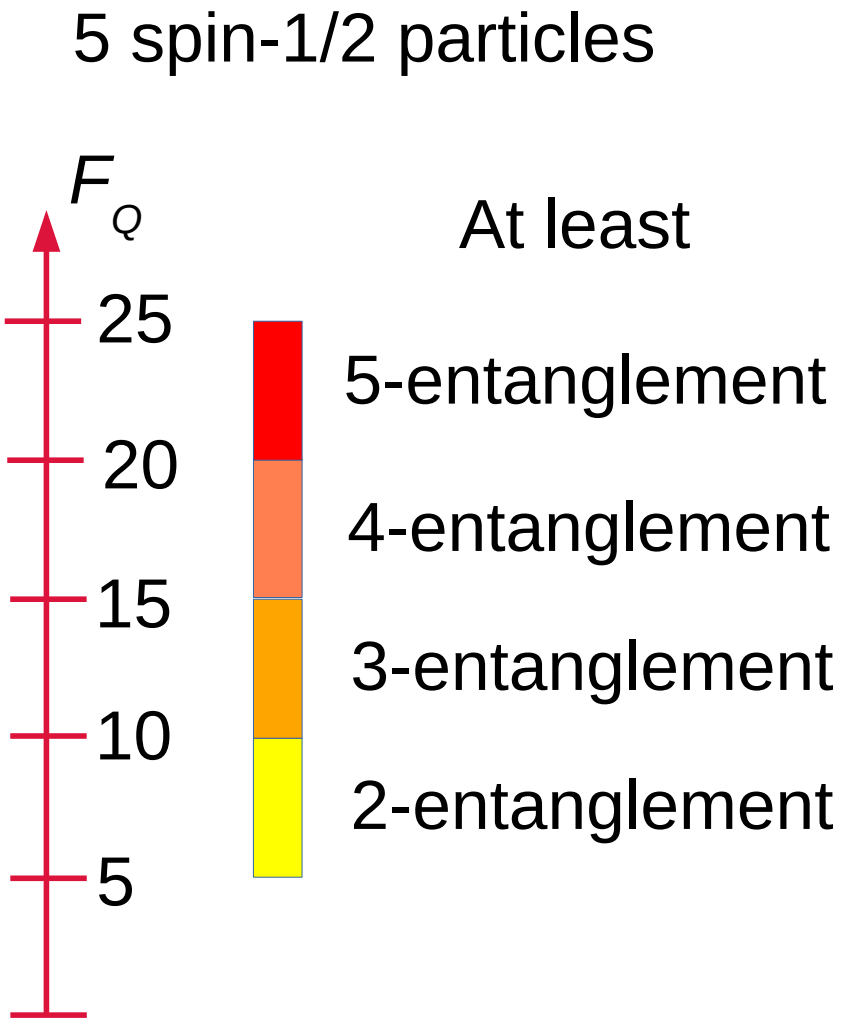
P. Hyllus *et al.* Phys. Rev. A 85, 022321 (2012);

GT, Phys. Rev. A 85, 022322 (2012).

- Bound for all quantum states of N qubits

$$F_Q[\varrho, J_l] \leq N^2.$$

The quantum Fisher information vs. entanglement



(For simplicity, we used $F_Q[\varrho, J_l] \leq kN$, which is not tight.)

Let us use the Cramér-Rao bound

- For separable states

$$(\Delta\theta)^2 \geq \frac{1}{\nu N}, \qquad l = x, y, z.$$

Pezze, Smerzi, Phys. Rev. Lett. 102, 100401 (2009);

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- For states with at most k-particle entanglement (tight bound if k is a divisor of N)

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Metrology with Dicke states

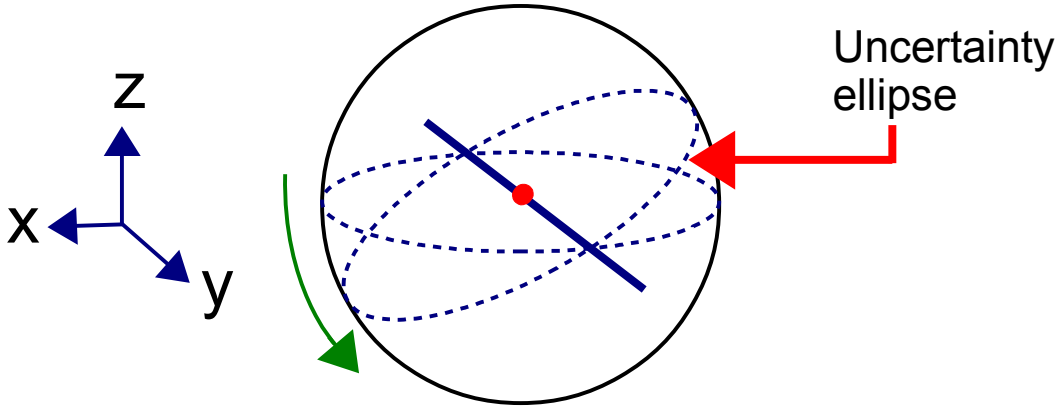
- For Dicke state

$$\langle J_l \rangle = 0, l = x, y, z, \quad \langle J_z^2 \rangle = 0, \quad \langle J_x^2 \rangle = \langle J_y^2 \rangle = \text{large}.$$

- Linear metrology

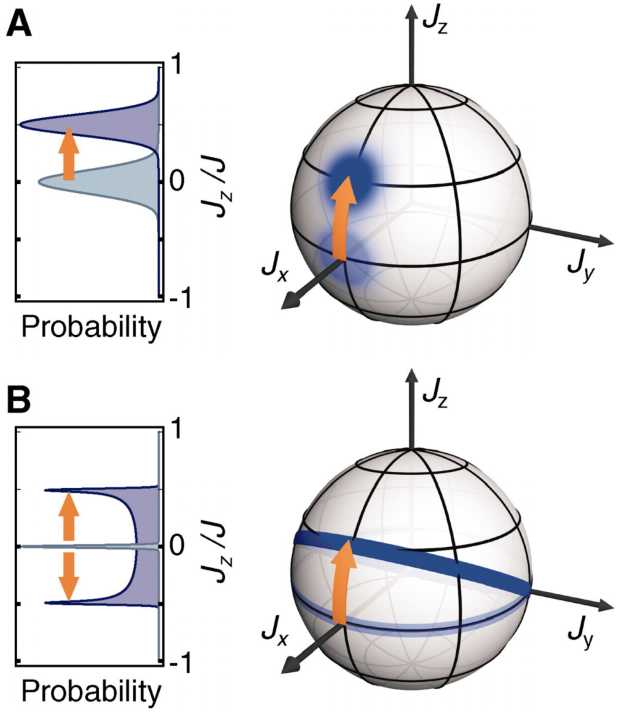
$$U = \exp(-iJ_y\theta).$$

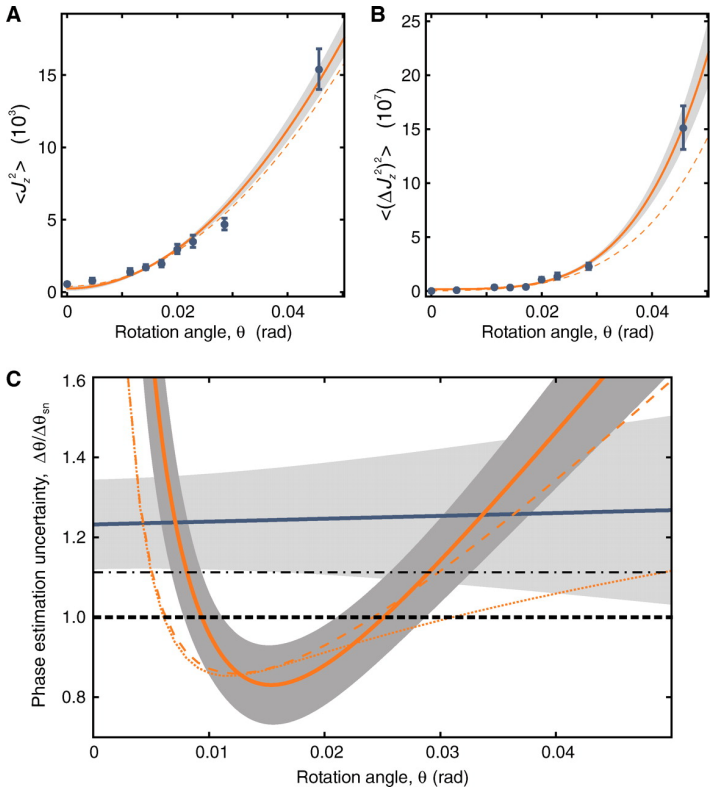
- Measure $\langle J_z^2 \rangle$ to estimate θ. (We cannot measure first moments, since they are zero.)



Metrology with Dicke states

Experiment with cold gas of 8000 atoms.





Lücke M. Scherer, Kruse, Pezzé, Deuretzbacher, Hyllus, Topic, Peise, Ertmer, Arlt, Santos, Smerzi, Klempt, Science 2011.

Multipartite entanglement vs. metrology

- Conclusion: Multipartite entanglement is needed for high precision quantum metrology.
- Without high level of multipartite entanglement, we cannot have high precision quantum metrology.

Thanks to:

- Theory: O. Gühne, Siegen U., Germany
- G. Vitagliano, IQOQI, Vienna, Austria
- I. Apellaniz, P. Hyllus, I. L. Egusquiza, I. Urizar-Lanz, UPV/EHU, Bilbao, Spain
- M. Fadel, ETH Zürich and Basel, Switzerland
- M. Kleinmann, Siegen U., Germany
- Z. Zimborás, Wigner, Hungary
- H.-J. Briegel, Insbruck, Austria
- L. Santos, Hannover U., Germany
- L. Pezze, A. Smerzi, LENS, Firenze
- Photonic experiments: H. Weinfurter, N. Kiesel, W. Wieczorek, L. Knips, MPQ München
- Dicke state experiments with BEC: C. Klempt, I. Kruse, J. Peise, K. Lange, B. Lücke, Hannover U., Germany; J. Arlt, Arhus U., Denmark
- Singlet state experiments with cold atoms: M. W. Mitchell, R. J. Sewell, N. Behbood, F. Martin Ciurana, G. Colangelo, M. Napolitano, J. Kong, R. Jiménez-Martínez, C. Troullinou, V. G. Lucivero, ICFO, Barcelona

Summary

- We discussed entanglement detection in particle ensembles.

THANK YOU FOR YOUR ATTENTION!

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