Alternatives of entanglement depth and metrological entanglement criteria

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Overview

- multipartite entanglement:
  classification / qualification / quantification

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$D(\rho)$ (prod.)

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\rotatebox{90}{$\geq$} & \\
D_{\text{avg}}(\rho) & \text{(avg.)}
\end{array}
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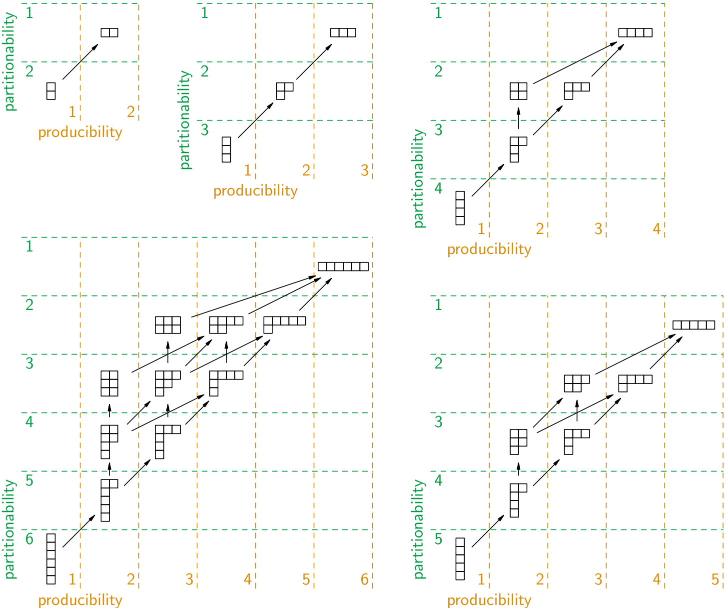
Szalay, Tóth, arXiv:2408.15350 **[quant-ph]** (2024), under review in Quantum

Partitionability and producibility



- height, width and Dyson-rank of a Young diagram

$$h(\hat{\xi}) := |\hat{\xi}|$$
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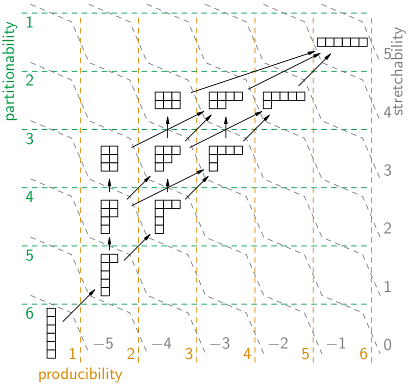

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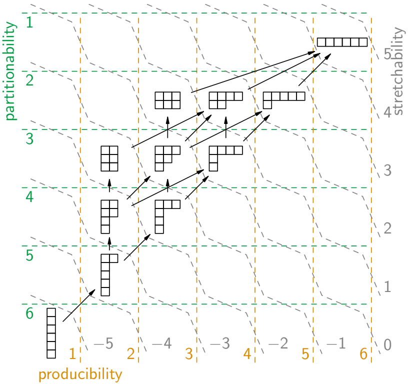

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- k-prod. states: $\mathcal{D}_{k\text{-prod}}$
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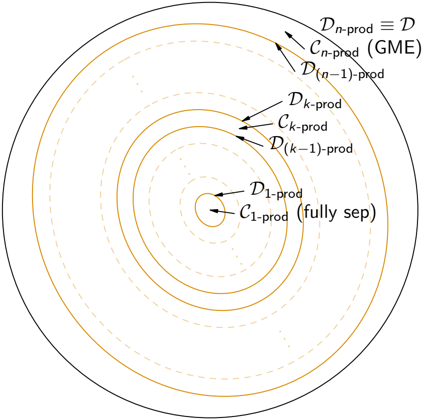

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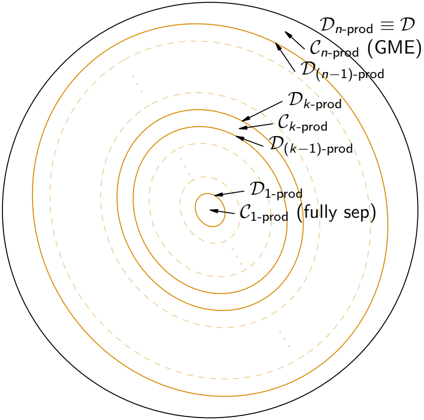
- depth of part., prod., and str.:



$$D_{\text{part}}(\rho) := \max\{k \in h(\hat{P}_{\text{I}}) \mid \rho \in \mathcal{D}_{k\text{-part}}\}$$
$$D_{\text{prod}}(\rho) := \min\{k \in w(\hat{P}_{\text{I}}) \mid \rho \in \mathcal{D}_{k\text{-prod}}\} \equiv D(\rho)$$
$$D_{\text{str}}(\rho) := \min\{k \in r(\hat{P}_{\text{I}}) \mid \rho \in \mathcal{D}_{k\text{-str}}\}$$

One-parameter entanglement properties, squareability

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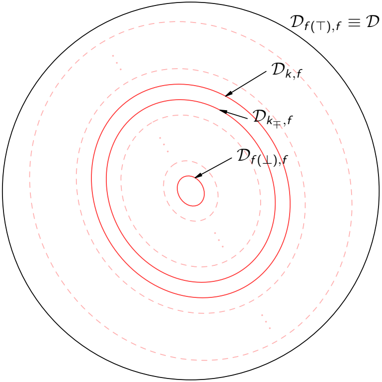

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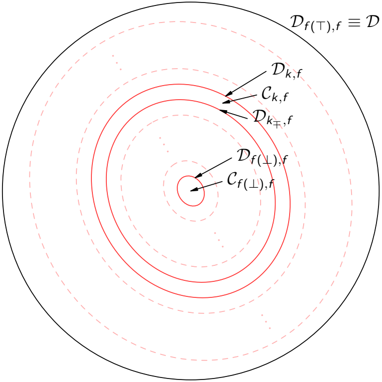

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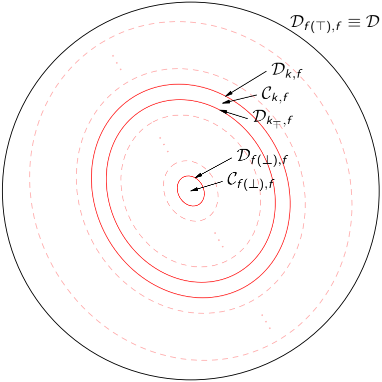

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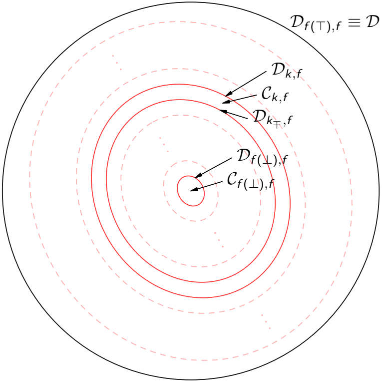

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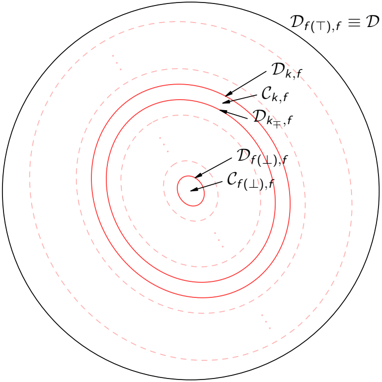

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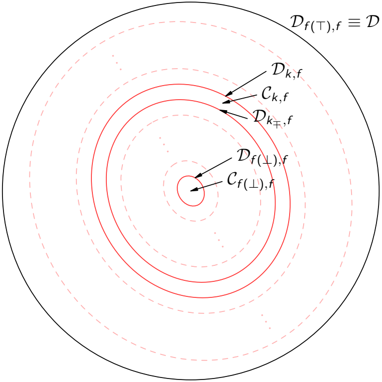

$$s_2(\hat{\xi}) = \sum_{x\in\hat{\xi}} x^2 = n \sum_{x\in\hat{\xi}} \frac{x}{n} x = n\,\mathrm{avg}(\hat{\xi})$$

average size of entangled subsystems (w.r.t. picking elementary subsys.)

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Convex metrological entanglement criteria

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- altogether we have

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average size of entangled subsystems (ASES)

Convex vs. original: $F_{\mathsf{Q}}(\rho, J^{\mathsf{z}})/n \leq D^{\mathsf{oF}}(\rho) \leq D(\rho)$

weaker bound $F_{\mathsf{Q}}(\rho, J^{\mathsf{z}})/n \leq D(\rho)$

- $\rho_\epsilon := \epsilon\pi_k + (1-\epsilon)\rho_1$ for $\epsilon > 0$
  with $\pi_k := |\psi_k\rangle\langle\psi_k| \in \mathcal{C}_{k\text{-prod}}$ and $\rho_1 \in \mathcal{C}_{1\text{-prod}}$, $\mathrm{Tr}(\pi_k\rho_1) = 0$

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- ρ_ϵ is much less entangled as π_k itself, a much lower F_Q/n is expected

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stronger bound $F_Q(\rho, J^z)/n \leq D^{oF}(\rho)$

- for all pure decompositions $\rho = \sum_i p_i \pi_i$, let $q_k = \sum_{i:D(\pi_i)=k} p_i$

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- or at least the same q_1 weight of 5-producible states $3 \leq 1q_1 + 3q_3 + 5q_5 = q_1 + 3(1 - q_1 - q_5) + 5q_5$ leads to $q_1 \leq q_5$

Take home message

One-parameter partial separability properties

- given by generator functions of partitions, monotones for refinement

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- metrological precision (by quantum Fisher Information) vs.
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- for example

$$\begin{array}{ccccccc} & & D^{\mathrm{oF}}(\rho) & \leq & D(\rho) & & \text{(prod.)} \\ & & \vee\!| & & \vee\!| & & \\ \frac{1}{n}F_{\mathrm{Q}}(\rho, J^z) & \leq & D^{\mathrm{oF}}_{\mathrm{avg}}(\rho) & \leq & D_{\mathrm{avg}}(\rho) & & \text{(avg.)} \end{array}$$

Thank you for your attention!

"Alternatives of entanglement depth and metrological entanglement criteria"

Szalay, Tóth, arXiv:2408.15350 **[quant-ph]** (2024), under review in Quantum

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