Activation of metrologically useful genuine multipartite entanglement

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Quantum Fisher information

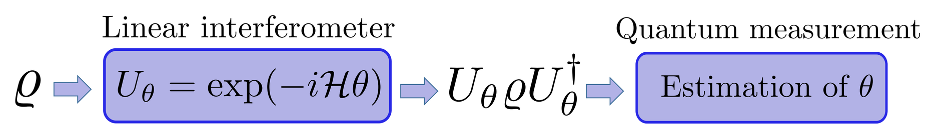

Figure 1: Typical process of quantum metrology

- $\mathcal{H}$ is assumed to be *local*, that is,

$$\mathcal{H} = h_1 + \cdots + h_N, \tag{1}$$

where h_n's act on single-subsystems.

- Cramér-Rao bound:

$$(\Delta\theta)^2 \geq 1/\mathcal{F}_Q[\varrho, \mathcal{H}], \tag{2}$$

where the quantum Fisher information is given by

$$\mathcal{F}_Q[\varrho, \mathcal{H}] = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} |\langle k|\mathcal{H}|l\rangle|^2, \tag{3}$$

with $\varrho = \sum_k \lambda_k |k\rangle\langle k|$ being the eigendecomposition. In general:

$$4(\Delta\mathcal{H})^2 \geq \mathcal{F}_Q[\varrho, \mathcal{H}] \geq 4I_\varrho(\mathcal{H}), \tag{4}$$

with $I_\varrho(\mathcal{H}) = \mathrm{Tr}(\varrho\mathcal{H}^2) - \mathrm{Tr}(\sqrt{\varrho}\mathcal{H}\sqrt{\varrho}\mathcal{H})$.

Metrological gain

- The metrological gain for a probe state ϱ and a Hamiltonian $\mathcal{H}$ is defined by [1]

$$g_\mathcal{H}(\varrho) = \mathcal{F}_Q[\varrho, \mathcal{H}]/\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}), \tag{5}$$

where for a given *local* Hamiltonian $\mathcal{H}$, separable states can achieve at most

$$\mathcal{F}_Q^{(\mathrm{sep})}(\mathcal{H}) = \sum_{n=1}^{N} [\sigma_{\max}(h_n) - \sigma_{\min}(h_n)]^2. \tag{6}$$

- $g_\mathcal{H}(\varrho)$ in Eq. (5) can be maximized over *local* Hamiltonians [1]

$$g(\varrho) = \max_{\mathrm{local}\,\mathcal{H}} g_\mathcal{H}(\varrho). \tag{7}$$

- A quantum state is *useful* for metrology if $g(\varrho) > 1$.
- Scaling propeties
  - Shot-noise scaling: for separable states $g_\mathcal{H} \sim 1$ ($\mathcal{F}_Q \sim N$) at best.
  - Heisenberg scaling: for entangled states $g_\mathcal{H} \sim N$ ($\mathcal{F}_Q \sim N^2$) at best.

The many copy scheme

- Quantum entanglement is required for metrological usefulness [2].
- But there are highly entangled pure states that are not useful [3], while weakly entangled bound entangled states can be useful [4, 5].
- Can entangled states be made useful with the idea of having more copies [6]? Can we have $g(\varrho^{\otimes M}) > g(\varrho)$?

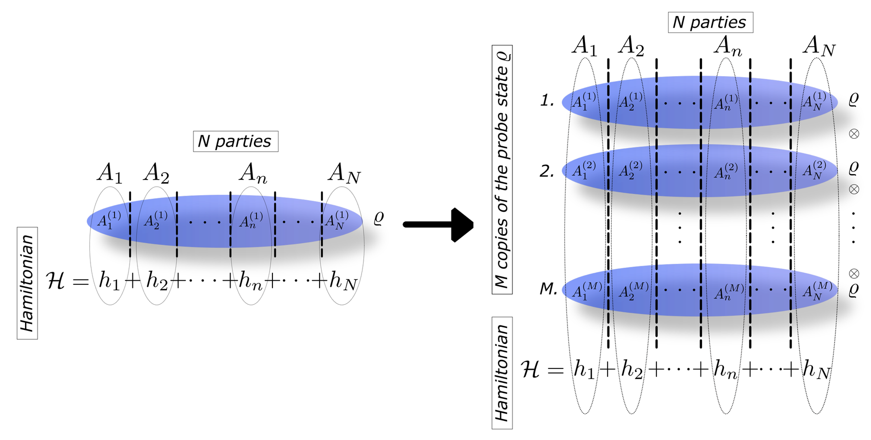

Figure 2: M copies of the N-partite state ϱ.

- Large class of entangled states become maximally useful in the limit of many copies.
- Non-useful states can be made useful by embedding into higher dimension.

Maximal usefulness

Entangled states of $N \geq 2$ qudits of dimension d are maximally useful in the infinite copy limit if they live in the subspace

$$\{|0..0\rangle, |1..1\rangle, ..., |d-1, .., d-1\rangle\}. \tag{8}$$

For the *proof*, use Eq. (4) and calculate $I_{\varrho^{\otimes M}}(\mathcal{H})$, where $h_n = (D^{\otimes M})_{A_n}$ with $D = \mathrm{diag}(+1, -1, +1, -1, ...)$ and

$$\varrho = \sum_{k,l=0}^{d-1} c_{kl} (|k\rangle\langle l|)^{\otimes N}. \tag{9}$$

- *Example*: $|\mathrm{GHZ}_N\rangle = \frac{(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})}{\sqrt{2}}$ with noise:

$$\varrho_p = p|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N| + (1-p)\frac{(|0\rangle\langle 0|)^{\otimes N} + (|1\rangle\langle 1|)^{\otimes N}}{2}. \tag{10}$$

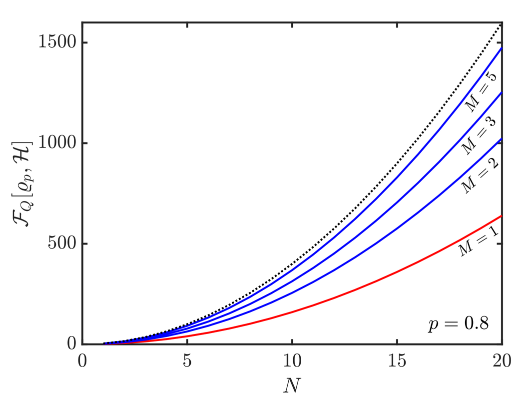

Figure 3: $\mathcal{F}_Q$ for different number of copies (M) of Eq. (10) as a function of the number of parties N with $p = 0.8$. The Hamiltonian is $h_n = \sigma_z^{\otimes M}$.

- *Example*: All entangled pure states of the form

$$\sum_{k=0}^{d-1} \sigma_k |k\rangle^{\otimes N}, \tag{11}$$

with $\sum_k |\sigma_k|^2 = 1$.

Embedding states

The state in Eq. (11) with $\sum_k |\sigma_k|^2 = 1$ is useful for $d \geq 3$ and $N \geq 3$.

- *Embedding into higher dimension*: The state

$$|\psi\rangle = \sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} \tag{12}$$

is useful if $1/N < 4|\sigma_0\sigma_1|^2$ [3]. But

$$\sigma_0 |0\rangle^{\otimes N} + \sigma_1 |1\rangle^{\otimes N} + 0\,|2\rangle^{\otimes N} \tag{13}$$

is always useful.

- *Example*: For $|\psi\rangle^{\otimes M}$ from Eq. (12) with $1/N = 4|\sigma_0\sigma_1|^2$:

$$\mathcal{F}_Q = 4N^2[1 - (1 - 1/N)^M]. \tag{14}$$

- *Example*: Embedding the noisy GHZ

$$\varrho_N^{(p)} = p|\mathrm{GHZ}_N\rangle\langle\mathrm{GHZ}_N| + (1-p)\mathbb{1}/2^N. \tag{15}$$

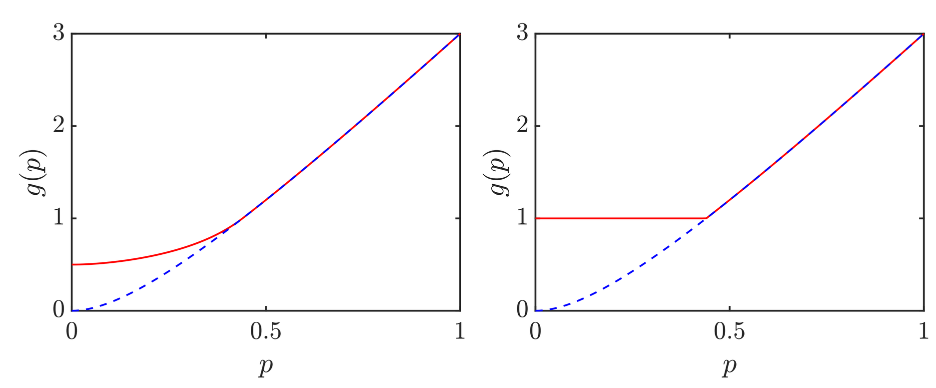

Figure 4: Embedding (solid) $\varrho_3^{(p)}$ into (left) $d = 3$, (right) $d = 4$.

$\varrho_3^{(p)}$ is metrologically useful if $p > 0.4396$ and genuine multipartite entangled if $p > 0.4286$.

Tolerating phase noise

More copies of a state can protect it from certain types of noise in a metrological task. In the following, we take $|\mathrm{GHZ}\rangle \equiv |\mathrm{GHZ}_3\rangle$.

- *Example*: Phase noise for $M = 1$ copy of the $|\mathrm{GHZ}\rangle$ state. The Hamiltonian is $\mathcal{H} = h_1 + h_2 + h_3$ with $h_n = \sigma_z$.

$$\mathcal{F}_Q[|\mathrm{GHZ}\rangle, \mathcal{H}] = 36 = 4N^2 \text{ (maximal)}, \quad \mathcal{F}_Q[\varrho, \mathcal{H}] < 36, \tag{16}$$

with the noisy state being

$$\varrho = p|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + (1-p)|\mathrm{GHZ}_\phi\rangle\langle\mathrm{GHZ}_\phi|,$$

where

$$|\mathrm{GHZ}_\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + e^{-i\phi}|111\rangle). \tag{17}$$

- *Example*: Tolerating phase noise for $M = 3$ copies of the $|\mathrm{GHZ}\rangle$ state. The Hamiltonian is $\mathcal{H} = h_1 + h_2 + h_3$ with $h_n = \sigma_z^{\otimes M}$.

$$\mathcal{F}_Q[|\mathrm{GHZ}\rangle^{\otimes 3}, \mathcal{H}] = 36 = 4N^2 \text{ (maximal)}, \quad \mathcal{F}_Q[\varrho, \mathcal{H}] = 36, \tag{18}$$

where ϱ is some mixture of states with phase-error on at most 1 copy:

$$|\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \quad |\mathrm{GHZ}_{\phi_1}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle, \quad |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_2}\rangle \otimes |\mathrm{GHZ}\rangle, \quad |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}\rangle \otimes |\mathrm{GHZ}_{\phi_3}\rangle. \tag{19}$$

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